Les diagrammes de Feynman, la partition du modèle standard

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L'intégrale de chemins

- Deux références utiles:
  - R.P. Feynman *Space-Time Approach to Non-Relativistic Quantum Mechanics* Review of Modern Physics 20, 367, (1948)
  - Cours de Claude C-T, *Forme lagrangienne de la mécanique quantique* http://www.phys.ens.fr/cours/notes-de-cours/cct-dea/index.html

- Reformulation par Feynman de la mécanique quantique
  - En mécanique quantique, la probabilité d'un événement pouvant se produire par plusieurs voies possibles est le carré du module de la somme contributions complexes associées, chacune, à l'une des voies possibles
  - La probabilité qu'une particule ait un chemin $x(t)$ dans une certaine région de l'espace-temps est le carré du module de la somme des contributions de tous les chemins possibles dans cette région
  - La contribution d'un chemin est postulée à valoir l'exponentielle dont la phase (imaginaire) est l'action classique (en unité de $\hbar$) le long de ce chemin
  - La contribution totale de tous les chemins atteignant x,t depuis le passé est la fonction d'onde $\psi(x,t)$
  - On démontre que cette fonction d'onde satisfait l'équation de Schrödinger

- **Rappels de mécanique classique**
  - **L'action classique** pour une particule à une dimension

$$S = \int_{t_a}^{t_b} \mathfrak{L}(\dot{x}, x, t)\,dt$$

$$\text{e.g.} \quad \mathfrak{L} = \frac{m}{2}\dot{x}^2 - V(x,t)$$

  - **Principe de moindre action**

$$\delta x(t_a) = \delta x(t_b) = 0$$

$$\delta S = S(\overline{x} + \delta x) - S(\overline{x}) = 0$$

$$S(x+\delta x) = S(x) + \int_{t_a}^{t_b} \left(\delta \dot{x} \frac{\partial \mathfrak{L}}{\partial \dot{x}} + \delta x \frac{\partial \mathfrak{L}}{\partial x} \right) dt$$

$$\delta S = \delta x \frac{\partial \mathfrak{L}}{\partial \dot{x}} \Bigg|_{t_a}^{t_b} - \int_{t_a}^{t_b} \delta x \left[\frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{x}} \right) - \frac{\partial \mathfrak{L}}{\partial x} \right] dt$$

- **Expression mathématique**
  - Fractionnement de l'intervalle temporel

$$N\varepsilon = t_b - t_a ; \varepsilon = t_{i+1} - t_i ; t_0 = t_a ; t_N = t_b ;$$

$$x_0 = x_a ; x_N = x_b$$

  - Analogie avec l'intégrale de Riemann

$$K(b,a) = \lim_{\varepsilon \to 0} \frac{1}{A} \iiint \dots \int \exp\left\{ \frac{i}{\hbar} S[b,a] \right\} \frac{dx_1}{A} \frac{dx_2}{A} \dots \frac{dx_{N-1}}{A}$$

$$S[b,a] = \int_{t_a}^{t_b} \mathfrak{L}(\dot{x}, x, t)\, dt$$

$$A = \left(\frac{2\pi i \hbar \varepsilon}{m} \right)^{1/2} \text{ si } \mathfrak{L} = \frac{m}{2} \dot{x}^2 - V(x,t)$$

$$K(b,a) \equiv \int \exp\left[\frac{i}{\hbar} S(b,a) \right] \mathfrak{D}x(t)$$

Le premier terme s'annule à cause des conditions aux limites, et en annulant le deuxième terme pour toute variation δx on obtient les **équations d'Euler-Lagrange**:

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{x}}\right) - \frac{\partial \mathfrak{L}}{\partial x} = 0$$

- Amplitude en mécanique quantique

Ré-interprétation de **l'expérience d'Young**

$$P(b,a) = \left|K(b,a)\right|^2$$

$$K(b,a) = \sum_{\forall x(t) \text{ de } a \text{ à } b} \Phi\left[x(t)\right]$$

Hypothèse: $$\Phi\left[x(t)\right] \propto \exp\left\{\frac{i}{\hbar} S\left[x(t)\right]\right\}$$

– Equation **intégrale** pour des événements se succédant dans le temps

$$S[b,a] = S[b,c] + S[c,a]$$

$$K[b,a] = \int \exp\left\{\frac{i}{\hbar} S[b,c] + \frac{i}{\hbar} S[c,a]\right\} \mathfrak{D}x(t)$$

$$x_c = x(t_c)$$

$$K[b,a] = \int_{x_c} \int_c^b \exp\left\{\frac{i}{\hbar} S[b,c]\right\} K[c,a] \mathfrak{D}x(t) dx_c$$

$$= \int_{x_c} K[b,c] K[c,a] dx_c$$

– La **fonction d'onde**: amplitude totale pour arriver au point *(x,t)* depuis le passé dans une situation non spécifiée

$$\psi(x,t) = \int_{-\infty}^{\infty} K[x,t;x',t'<t] \psi(x',t') dx'$$

$$|\psi(x,t)|^2 = \Pr\{\text{particule en } x,t\}$$

- Avantages de cette reformulation
  - Relation classique/quantique clarifiée: à la limite où $\hbar$ tend vers 0 seul compte le chemin classique
  - La formulation dans l'espace-temps facilite le passage à la physique relativiste
  - Point de vue plus global: amplitude de probabilité associée à une histoire entière
  - Méthode applicable, à tout système dont l'équation classique découle d'un principe variationnel, par exemple le champ électromagnétique
  - Prépare le passage de la mécanique quantique à la théorie quantique des champs

– Interprétation physique de la dualité onde/corpuscule en physique quantique: analogie entre
  - L’optique géométrique (temps de parcours de la lumière et principe de Fermat) et la mécanique classique (action et principe de Hamilton), d’une part
  - L’optique ondulatoire (période de l’onde et principe de Huygens) et la mécanique quantique (constante de Planck $\hbar$ et fonction d’onde), d’autre part

– L’intégrale de chemins est une intégrale fonctionnelle (intégrale à une infinité continue de variables d’intégration), mais si l’integrand est une gaussienne ou le produit d’une gaussienne par un polynôme, l’intégrale peut être calculée analytiquement

Intégrale de chemins et théorie quantique des champs

The perturbative expansion in QFT

Academic case of a single self coupled neutral scalar field:
generating functional of the perturbative expansion

$$\mathfrak{K} = \int \mathfrak{D}\phi \exp\left\{\frac{i}{\hbar}\int d^4x \mathfrak{L}\big(\phi(x), \partial_\mu \phi(x)\big)\right\}$$

$$\mathfrak{L} = \mathfrak{L}_0 + j\phi$$

$j =$ source of field ϕ

$$\mathfrak{L}_0 = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + g\phi^r$$

Generating functional: expansion in powers of *j*;
Perturbative expansion: expansion in powers of *g*

– Feynman's rules and diagrams

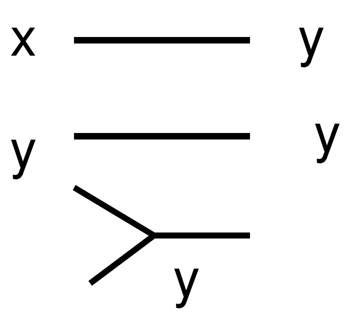
- In configuration space

The **Green's functions** are the coefficients of the expansion in powers of j

For N sources (**N point function**), the **perturbative expansion** is an expansion in powers of g : the coefficient of g^n is a sum of **Feynman amplitudes** associated each, by means of **Feynman rules** to a **Feynman diagram** with n **vertices**.

Vertices are the space time points where the r interacting fields are coupled.

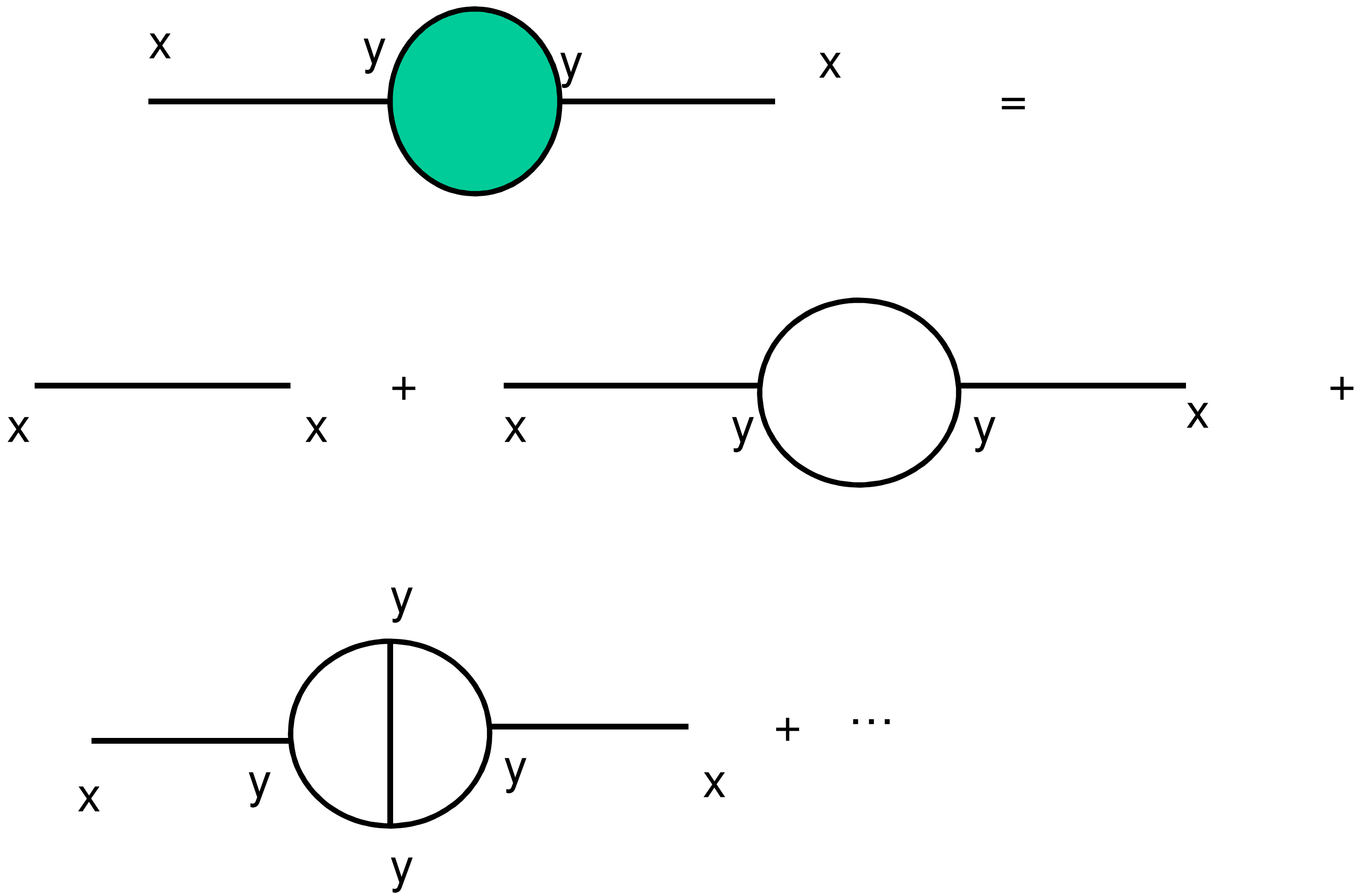
Internal lines relating two vertices, and **external** relating a vertex to a source are **propagators** corresponding to **free particle Green's functions**



N external lines

I internal lines

N vertices with *r* incident lines


x
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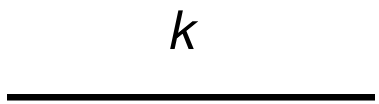
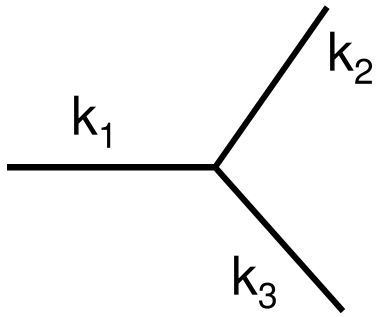
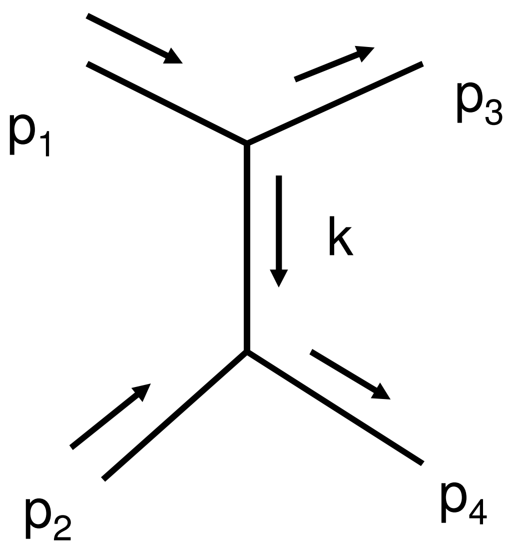
- Feynman rules (in momentum space):integration over vertices Y



$$\frac{1}{k^2 - m^2}$$



$$g\delta(k_1 + k_2 + k_3)$$

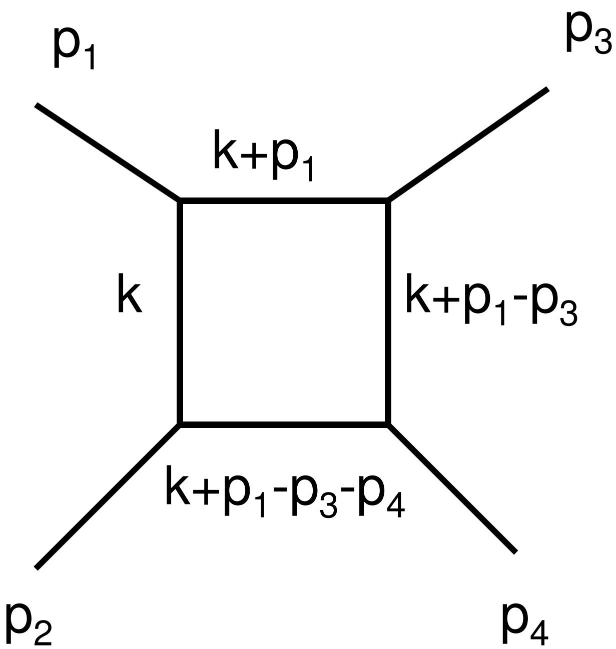


Born approximation

$$A^{(B)} = \frac{g^2}{k^2 - m^2} = \frac{g^2}{\left(p_1 - p_3\right)^2 - m^2} = \frac{g^2}{t - m^2}$$

Fourier transform of $V(r) = g^2 \frac{\exp(-mr)}{r}$

Loop integral

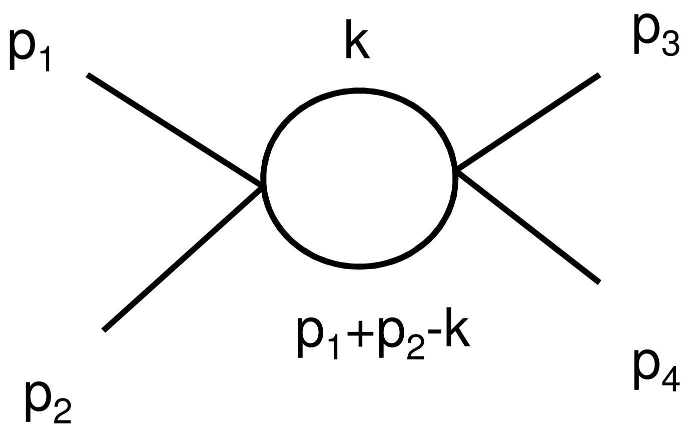


In the considered case, the integral converges; but what happens when $r = 4$?

Quantum fluctuation

(expansion in powers of $\hbar$)

$$A^{\text{loop}} = g^4 \int d^4k \frac{1}{\left(k^2 - m^2\right)\left((k+p_1)^2 - m^2\right)\left((k+p_1-p_3)^2 - m^2\right)\left((k+p_1-p_3-p_4)^2 - m^2\right)}$$



The loop integral diverges. **One needs to renormalise!**

$$r = 4$$

$$A^{\text{loop}} = g^2 \int d^4k \frac{1}{\left(k^2 - m^2\right)\left((p_2 + p_2 - k)^2 - m^2\right)}$$

L'intégrale de chemins en QED

– Équation de **Dirac** et courant conservé

- Lagrangien de Dirac

$$\mathfrak{L}_{\text{Dirac}} = \frac{i}{2}\left[\bar{\psi}\gamma^{\mu}\left(\partial_{\mu}\psi\right) - \left(\partial_{\mu}\bar{\psi}\right)\gamma^{\mu}\psi\right] - m\bar{\psi}\psi$$

$$\equiv \bar{\psi}(x)(i\gamma^{\mu}\partial_{\mu} - m)\psi(x)$$

- Équation de Dirac et possible **courant conservé**

$$\left(i\gamma.\vec{\partial} - m\right)\psi \equiv \left(i\partial\!\!\!/ - m\right)\psi = 0 ; \bar{\psi}\left(i\gamma.\overleftarrow{\partial} + m\right) = 0$$

$$\Rightarrow \partial_{\mu} j^{\mu} = 0 \text{ si } j^{\mu} = q\bar{\psi}(x)\gamma^{\mu}\psi(x)$$

– Le lagrangien de QED

$$\mathfrak{L}_{\text{QED}} = \mathfrak{L}_{\text{Maxwell}} + \mathfrak{L}_{\text{Dirac}} + \mathfrak{L}_{\text{Interaction}}$$

$$\mathfrak{L}_{\text{Maxwell}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} ; F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$$

$$\mathfrak{L}_{\text{Dirac}} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi$$

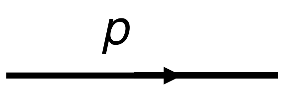
$$\mathfrak{L}_{\text{Interaction}} = -e\bar{\psi}\gamma^{\mu}A_{\mu}\psi$$

– Generating functional

$$\mathfrak{K}(j_{\psi}, j_{\bar{\psi}}, j_A) = \int \mathfrak{D}\psi\,\mathfrak{D}\bar{\psi}\,\mathfrak{D}A \exp\left\{\frac{i}{\hbar}\int d^4x(\mathfrak{L}_{\text{QED}} + j_{\psi}\psi + j_{\bar{\psi}}\bar{\psi} + j_A A)\right\}$$

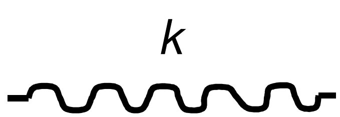
– Feynman's diagrams and rules

- Electron propagator



$$\frac{\not{p} + m}{p^2 - m^2}$$

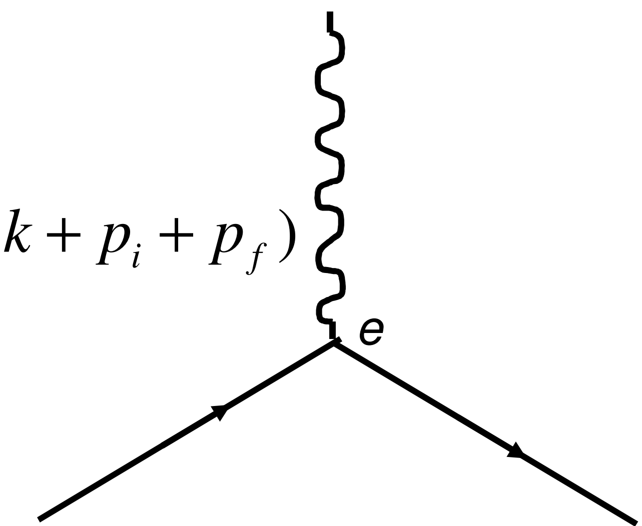
- Photon propagator



$$\frac{1}{k^2}$$

- Interaction vertex

$$\gamma^{\mu}\delta(k + p_i + p_f)$$

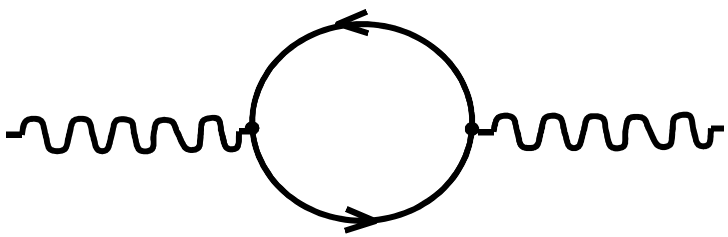


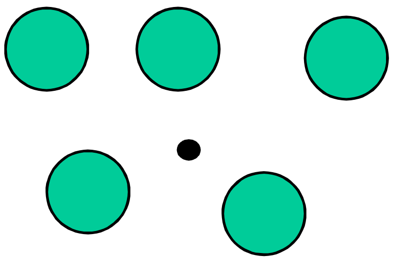
The divergence problem

- All Feynman amplitudes defined in terms of **ordinary integrals**
- In general, integrals implied by diagrams involving loops **diverge**
- These divergences reflect the **conflict** between **locality** (i.e. relativity + causality) and **quantum mechanics**: quantum fields are operator valued **distributions whose products at the same point is ill-defined**
- In QED, all divergences are **logarithmic**

Quantum vacuum as a dielectric: a heuristic picture of renormalisation

– Quantum fluctuations and vacuum polarization.
  - In the Fock space of the electromagnetic field, vacuum is the state with 0 photon. According to Heisenberg's inequalities, when the number of quanta is determined (0 in the vacuum) the value of the field is completely undetermined; it fluctuates
  - One such fluctuation corresponds to the **appearance followed by the disappearance of an electron-positron pair**
  - These fluctuations produce a **screening effect**, leading to the notion of an **effective charge depending on the resolution**

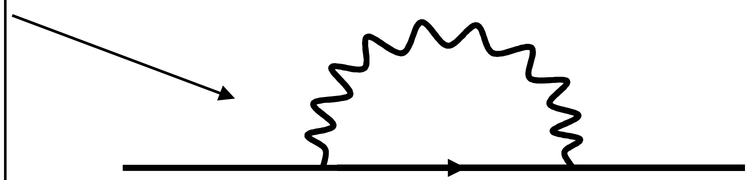




Effect of the interactions on the parameters of the theory

- The charge and the mass of the electron (the parameters of the theory) are **affected by the interaction with the electromagnetic field**.
- **self energy diagram**

An electron emits and reabsorbs a photon: the electron quantum field is modified by the electromagnetic fields it generates



- The parameters of the theory are split into "**bare**" parameters and "**dressed**" parameters, i.e. "**renormalized**" by the interaction
- Renormalized parameters are **effective parameters depending on the resolution**, one says that they are **renormalized at scale μ**

The "miracle" of Renormalizability

- **Renormalization**= redefinition of the fundamental parameters of the theory
  - Electric **charge** *e* split into
    - e_0 "**bare charge**", i.e. the charge if there were no interaction
    - *e* physical, "**renormalized**" charge
  - Electron **mass** *m* split into
    - m_0 "**bare mass**", i.e. the mass if there were no interaction
    - *m* physical, "**renormalized**" mass
- **Regularization**: With an energy **cutoff** Λ one removes the divergences in the expressions of *e, m* and any amplitude in terms of e_0 and m_0.

- One can **invert** these expressions

$$e_0 = e + 1/2\beta_2 e^3 \,\mathrm{Log}\left(\frac{\Lambda}{m}\right) + O(e^5)$$

$$m_0 = m + \gamma_1 m e^2 \,\mathrm{Log}\left(\frac{\Lambda}{\mathrm{m}}\right) + O(e^4)$$

and express **all amplitudes in terms of *e* and *m***

- The "**miracle of renormalizability**" is then that in these expressions, for any amplitude, and at all orders of perturbation expansion, **the limit when Λ goes to infinity is finite**, which means that one can express perturbatively all observables **in terms of two and only two physical parameters** which can be measured experimentally.
- The theory/experiment agreement is **amazingly good**

For example, for the magnetic moment of the electron one finds

Exp: 2,00231930482+/- 40
Th: 2,00231930476+/- 52

The renormalization group

- If the electron mass *m* was equal to zero, it would have been necessary to introduce a new energy scale μ, called the **renormalization energy**, to deal with the logarithmic divergence,

$$e_0 = e + 1/2\beta_2 e^3 \,\mathrm{Log}\left(\frac{\Lambda}{\mu}\right) + O(e^5)$$

and when one inverts this relation, *e* becomes a function of μ

- When $m \neq 0$

a renormalization energy is still necessary, but it is implicitly set equal to *m* : $e(\mu = m) = e\,; m(\mu = m) = m$

- The renormalization energy being arbitrary, physics must be independent on its value, which means that the renormalized charge must obey a **differential equation, called the renormalization group equation**, expressing this independence:

$$\mu \frac{de^2(\mu)}{d\mu} = \beta\left(e^2(\mu)\right)$$

where $\beta(e^2) = \beta_2 e^4 + O(e^6)$

is finite when Λ goes to infinity (β_2 is a c-number, depending only on the QED Lagrangian)

- As a consequence, parameters in QED have to be considered as **effective parameters**, depending on the resolution

$$\alpha(\mu = m_e) = 1/137 \; ; \alpha(\mu = m_Z) = 1/128$$

Gauge invariance

- In Maxwell theory, **gauge invariance** $(A_\mu \rightarrow A_\mu + \partial_\mu \Lambda(x))$ **is equivalent to current conservation**
- In QED, the following transformation (**local gauge invariance**) leaves the QED Lagrangian invariant

$$\psi(x) \rightarrow e^{-i\alpha(x)}\psi(x)$$

$$\overline{\psi(x)} \rightarrow e^{+i\alpha(x)}\overline{\psi(x)} \qquad \alpha(x) = e\Lambda(x)$$

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda(x)$$

- Conversely, one can impose invariance through a **local** change of the phase of the electron field by introducing a gauge invariant field, the electromagnetic field: **local gauge invariance determines the structure of the interaction**

Le modèle standard

Standard model and gauge theories

- Renormalisability of gauge theories (abelian or and non-abelian) with or without symmetry breaking ('t Hooft, Veltman, Zinn-Justin, Lee)
- Non abelian gauge theories
  - QCD $SU(3)_{color}$
    - Asymptotic freedom
    - Confinement
  - Electroweak theory SU(2) X U(1)
    - Higgs mechanism
    - Masses of intermediate vector bosons and of fermions
- Excellent agreement theory / experiment up to 200 GeV
- Missing link: the Higgs boson. LHC 2010(?)

Coarse grain history of the Standard Model

Dates	Cadre théorique	Gravitation	Électro magnétisme	Interaction faible	Interaction forte
17ème siècle	Galilée, Newton	Newton			
19ème siècle	Euler, Lagrange, Jacobi, Hamilton		Maxwell		
1895-1898			Rayons X, électron, radioactivité		
1900-1930	Mécanique quantique				
1905-1915	Relativité	Einstein			
1930-1950	Théorie quantique des champs		QED	Fermi	Yukawa
1970-2000	Théories de jauge	Big bang	Théorie électrofaible de Glashow, Salam et Weinberg		QCD
2003- …	Supersymétrie Supercordes, Gravitation quantique?	Inflation	MSSM ?		

Fundamental fermions

Generation / Type	1st generation	2nd generation	3rd generation
q=2/3 quarks	**Up** u (W EM S)	**Charm** c (W EM S)	**Top** t (W EM S)
q=-1/3 quarks	**Down** d (W EM S)	**Strange** s (W EM S)	**Beauty** b (W EM S)
Neutral leptons (neutrinos)	**Electron neutrino** ν_e (W)	**Muon neutrino** ν_μ (W)	**Tauon neutrino** ν_τ (W)
Charged leptons	**Electron** e (W EM)	**Muon** μ (W EM)	**Tauon** τ (W EM)

Fundamental interactions

Interaction	Involved particles	Charge	Boson
Strong	Quarks	Color	Gluons
Electromagnetic	Quarks, charged leptons	Electric charge	Photon
Weak	Quarks, charged leptons and neutrinos	Weak Isospin	Vector intermediate bosons, W^+, W^- Z^0
Gravity	All particles	Energy	Graviton