# Classical gauge field theory

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## ☐ References

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## ☐ Introduction

### ☐ Classical field theories

Consider as an example the free particle Klein-Gordon equation [1] $(\partial_\mu\partial^\mu + m^2)\phi = 0$ which follows from the conservation equation $p_\mu p^\mu - m^2 = 0$ with the

1. We forget about factors $\hbar$ and $c$ in this chapter.

correspondence $p_\mu \to i\partial_\mu$. We use the convention $g_{\mu\nu} = \text{diag}\,(+1,-1,-1,-1)$ for the metric tensor. For later use, we choose the MKSA conventions, e.g. $p^\mu = (E/c, \vec{p}) \equiv (E, \vec{p})$ and $A^\mu = (\phi/c, \vec{A}) \equiv (\phi, \vec{A})$, where an over-arrow denotes a vector in ordinary space, or $\partial^\mu = (\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}) \equiv (\frac{\partial}{\partial t}, -\vec{\nabla})$. Most of the time, $\hbar$ and $c$ are fixed to unity.

The Lagrangian density from which this equation follows must satisfy [2]

$$\frac{\delta}{\delta\phi^*}\int d^4x\,\mathcal{L}(\phi,\phi^*,\partial_\mu\phi,\partial_\mu\phi^*) = \frac{\partial\mathcal{L}}{\partial\phi^*} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} = 0, \tag{1}$$

$$\frac{\partial\mathcal{L}}{\partial\phi^*} = -m^2\phi, \tag{2}$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} = \partial_\mu\partial^\mu\phi. \tag{3}$$

The first condition is fulfilled if

$$\mathcal{L} = -m^2\phi^*\phi + \text{terms in } \partial_\mu\phi^*\partial^\mu\phi, \tag{4}$$

and the second if [3]

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \text{terms in } \phi^*\phi. \tag{5}$$

Eventually the Klein-Gordon Lagrangian is given by

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi. \tag{6}$$

The first term is usually referred to as kinetic energy, although the space part [4] is reminiscent from local interactions in the context of classical field theory, and the second term to the mass term, since it corresponds to the mass of the particles after quantization. The quantization procedure, not discussed here, consists in the promotion of the classical fields into creation (or annihilation) field operators which obey, together with the corresponding conjugate momenta, to canonical commutation relations. We will stay here at the level of

2. We consider complex scalar fields. For real fields, factors of $\frac{1}{2}$ would appear here and there.

3. We develop to obtain $\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} = \partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0\phi^*)} + \partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i\phi^*)} = \partial_\mu\partial^\mu\phi = (\partial_0\partial^0 + \partial_i\partial^i)\phi = (\partial_0\partial_0 - \partial_i\partial_i)\phi$, the solution $\mathcal{L} = \partial_0\phi^*\partial_0\phi - \partial_i\phi^*\partial_i\phi = \partial_0\phi^*\partial^0\phi + \partial_i\phi^*\partial^i\phi = \partial_\mu\phi^*\partial^\mu\phi$ (up to terms in $\phi^*\phi$) follows.

4. We have a space and a time part in $\partial_\mu\phi\partial^\mu\phi = \frac{1}{c^2}\left|\frac{\partial\phi}{\partial t}\right|^2 - |\vec{\nabla}\phi|^2$, and only the time derivatives are reminiscent of a kinetic energy.

classical field theory, which means that although the fields correspond to the wave functions of quantum objects in the first-quantized form of the theory, they are treated by classical field theory (e.g. Euler-Lagrange equations) and no quantum fluctuations are allowed.

### □ The idea behind non-Abelian gauge theory

According to Salam and Ward, cited by Novaes in hep-ph/0001283 :

*Our b sic postul te is th t it should be possible to gener te strong, we k, nd electrom gnetic inter ction terms ... by m king loc l g uge tr nsform - tions on the kinetic-energy terms in the free L gr ngi n for ll p rticles.*

or Yang and Mills cited in A.C.T. Wu and C.N. Yang, Int. J. Mod. Phys. Vol. 21, No. 16 (2006) 3235 :

*The conserv tion of isotopic spin points to the existence of fund ment l inv ri nce l w simil r to the conserv tion of electric ch rge. In the l tter c se, the electric ch rge serves s source of electrom gnetic field. An import nt concept in this c se is g uge inv ri nce which is closely connected with (1) the equ tion of motion of the electrom gnetic field, (2) the existence of current density, nd (3) the possible inter ctions between ch rged field nd the electrom gnetic field. We h ve tried to gener lize this concept of g uge inv ri nce to pply to isotopic spin conserv tion.*

### □ The origin of gauge invariance[5]

The idea of "gauging" a theory, i.e. making local the symmetries, is due to E. Noether, but gauge invariance was introduced by Weyl when he tried to incorporate electromagnetism into geometry through the idea of local scale transformations. From one point of space-time to an other at a distance $dx^\mu$, the scale is changed from 1 to $(1 + S_\mu dx^\mu)$ in such a way that a space-time dependent function (of dimension of a length) $f(x)$ is changed according to

$$f(x) \to f(x+dx) = (f + \partial_\mu f dx^\mu))(1 + S_\mu dx^\mu) \simeq f + [(\partial_\mu + S_\mu)f]dx^\mu. \quad (7)$$

The original idea of Weyl was to identify $S_\mu$ to the 4−potential $A_\mu$, but with the advent of quantum mechanics and the correspondence between $p_\mu$ and $i\partial_\mu$, it was later realized that the correct identification is $S_\mu \leftrightarrow iqA_\mu$. Weyl nevertheless retained his original terminology of *g uge inv ri nce* as an invariance under a change of length scaled, or a change of the gauge.

5. See Cheng and Li pp235-236, and Gross p956

# □ Abelian $U(1)$ gauge theory

## □ The complex nature of the field as an internal structure

The prototype of gauge theory is the theory of electromagnetism, or Abelian $U(1)$ theory, where charge conservation is deeply connected to global phase invariance in quantum mechanics (a connection probably made first by Hermann Weyl).

The complex scalar field $\varphi(x)$ (Schrödinger or Klein-Gordon field) can be modified by a global phase transformation $\varphi(x) \to \exp(iq\alpha)\varphi(x)$ (Abelian $U(1)$ gauge transformation) which leaves the matter Lagrangian $\mathcal{L} = \partial_\mu\varphi^*\partial^\mu\varphi - V(|\varphi|^2)$ unchanged. We anticipate and introduce already the charge $q$ which couples the particle to the electromagnetic field[6]. Let us write $\varphi = \phi^1 + i\phi^2$ and introduce a real two-component field $\boldsymbol{\phi}(x) = \begin{pmatrix}\phi^1\\ \phi^2\end{pmatrix}$. The gauge transformation now appears as an Abelian rotation in a two-dimensional (internal) space.

## □ Noether theorem and matter current density

Consider the Langrangian density

$$\mathcal{L}_0 = \partial_\mu\varphi^*\partial^\mu\varphi - V(\varphi^*\varphi). \tag{8}$$

For later use, we will call this Lagrangian the function

$$\mathcal{L}_0 = F(\varphi, \varphi^*, \partial_\mu\varphi, \partial_\mu\varphi^*). \tag{9}$$

For any symmetry transformation,

$$\delta\mathcal{L}_0 = \delta\varphi^*\frac{\partial\mathcal{L}_0}{\partial\varphi^*} + \delta(\partial_\mu\varphi^*)\frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\varphi^*)} + \varphi^* \to \varphi = 0. \tag{10}$$

The notation $\varphi^* \to \varphi$ means that we add a similar term with all complex numbers replaced by their conjugate. The global phase transformation $\varphi \to \varphi' = \varphi e^{iq\alpha}$ being such a symmetry, we put $\delta\varphi = iq\alpha\varphi$ and $\delta(\partial_\mu\varphi) = iq\alpha\partial_\mu\varphi$

6. Here we set $\hbar = 1$. In most of the relations, $\hbar$ is restored through the substitution $q \to q/\hbar$, $g \to g/\hbar$, or $i\partial_\mu \to i\hbar\partial_\mu$.

in the expression of $\delta\mathcal{L}_0$ to have

$$\begin{aligned}\delta\mathcal{L}_0 &= -iq\alpha\left[\varphi^*\frac{\partial\mathcal{L}_0}{\partial\varphi^*}+\partial_\mu\varphi^*\frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\varphi^*)}\right]+\varphi^*\to\varphi\\ &= -iq\alpha\partial_\mu\left[\varphi^*\frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\varphi^*)}\right]+\varphi^*\to\varphi\\ &= -\alpha\partial_\mu j^\mu. \end{aligned}\tag{11}$$

It follows that the Noether current

$$j^\mu = -iq\left[\varphi\frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\varphi)}-\varphi^*\frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\varphi^*)}\right]\tag{12}$$

is conserved, $\partial_\mu j^\mu = 0$. The electric charge conservation thus appears as the consequence of the invariance of the theory under global phase changes, this is called a global gauge symmetry. Note that in the case of the Klein-Gordon Lagrangian, the conserved current takes the form[7]

$$j^\mu = -iq(\varphi\partial^\mu\varphi^*-\varphi^*\partial^\mu\varphi).\tag{13}$$

### □ Local gauge symmetry

Extending the gauge symmetry to local transformations requires the introduction of a (vector) gauge field which will be seen later as the vector potential of electromagnetism. In other words, making local the gauge symmetry builds the electromagnetic interaction.

Let us assume that the gauge transformation is *loc l*, i.e. $\varphi\to\varphi' = \varphi e^{iq\alpha(x)}\equiv G(x)\varphi$. We note that now $\partial_\mu\varphi\to\partial_\mu\varphi' = G(x)\partial_\mu\varphi+(\partial_\mu G(x))\varphi\neq G(x)\partial_\mu\varphi$, and the derivative of the field does not transform like the field itself does. Let us define the *cov ri nt* derivative[8]

$$\mathcal{D}_\mu\equiv\partial_\mu+iqA_\mu\tag{14}$$

where $A_\mu$ is still to be defined by its transformation properties. Like $\varphi' = G(x)\varphi$, we demand that

$$\mathcal{D}_\mu\varphi\to(\mathcal{D}_\mu\varphi)'\equiv G(x)\mathcal{D}_\mu\varphi.\tag{15}$$

7. Note that at this point, the sign in front of the current density was arbitrary, but if one wants to recover the usual expression of probability density current in quantum mechanics, e.g. in the Schrödinger case, $\vec{j} = \frac{\hbar}{2mi}\varphi^*\vec{\nabla}\varphi+\varphi^*\to\varphi$, it leads to the present charge current density after being multiplied by $q$.

8. The term covariant refers to covariance with respect to the introduction of the local transformation, and not to covariant-contravariant indices.

Since we have $(\mathcal{D}_\mu\varphi)' = (\partial_\mu\varphi)' + iqA'_\mu\varphi' = G(x)\partial_\mu\varphi + (\partial_\mu G(x))\varphi + iqA'_\mu G(x)\varphi$ and $G(x)\mathcal{D}_\mu\varphi = G(x)\partial_\mu\varphi + iqA_\mu G(x)\varphi$, we must require that $iqA'_\mu G(x) = iqA_\mu G(x) - \partial_\mu G(x)$ or, multiplying by $G^{-1}(x)$,

$$A'_\mu = A_\mu + \frac{i}{q}G^{-1}(x)\partial_\mu G(x) = A_\mu - \partial_\mu\alpha(x). \tag{16}$$

Demanding the invariance property of the kinetic term (according to Salam and Ward cited above) in the Lagrangian density under a local gauge transformation requires the introduction of a vector field which obeys the usual transformation law of the vector potential of electromagnetism through gauge transformations. These transformations which appeared before as a kind of mathematical curiosity of Maxwell theory are now necessary in order to preserve local gauge invariance. In a sense, the interaction is "created" by the principle of local gauge symmetry, while the principle of global gauge symmetry implies the conservation of the electric charge.

The interaction of matter (as described by the Lagrangian density $\mathcal{L}_0$) with the electromagnetic field can be built in through essentially two different approaches. In a first approach, we successively add terms to $\mathcal{L}_0$ in order to get at the end a locally gauge invariant Lagrangian. Since the prescription of local gauge invariance induces Maxwell interactions, this should automatically incorporate interaction terms in $\mathcal{L}$. Starting with the observation that $\mathcal{L}_0$ is no longer gauge invariant through *loc l* transformations, since $\delta\mathcal{L}_0 = -\alpha(x)\partial_\mu j^\mu - (\partial_\mu\alpha(x))j^\mu$ now contains the second term, we have to kill this last term by the introduction of a $\mathcal{L}_1 = -j_\mu A^\mu$ term [9]. This again generates one more contribution in $\delta(\mathcal{L}_0 + \mathcal{L}_1)$ which is canceled if we add $\mathcal{L}_2 = q^2 A_\mu A^\mu\varphi^*\varphi$. The combination $\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$ is now locally gauge invariant.

In a shorter approach, called *minim l coupling*, we simply replace the kinetic term $\partial_\mu\varphi^*\partial^\mu\varphi$ in $\mathcal{L}_0$ by $(\mathcal{D}_\mu\varphi)^*(\mathcal{D}^\mu\varphi)$ which was especially constructed in order to be gauge invariant (under *loc l* gauge transformations). One also has to add the pure field contribution $\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ to get the full Lagrangian

9. Note that here $j^\mu$ is the current which *was conserved in the absence of gauge interaction*. The sign here is also coherent with the expression $-(\rho\phi - \vec{j}\vec{A})$.

density [10]

$$\begin{aligned}\mathcal{L}_{\text{tot}} &= \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_A \\ &\equiv F(\varphi, \varphi^*, (\mathcal{D}_\mu\varphi), (\mathcal{D}_\mu\varphi)^*) + \mathcal{L}_A \\ &= (\mathcal{D}_\mu\varphi)^*(\mathcal{D}^\mu\varphi) - V(\varphi^*\varphi) + \mathcal{L}_A. \end{aligned} \tag{17}$$

The interaction terms are recovered from the r.h.s. (here in the Klein-Gordon case) :

$$\begin{aligned}(\mathcal{D}_\mu\varphi)^*(\mathcal{D}^\mu\varphi) &= (\partial_\mu\varphi + iqA_\mu\varphi)^*(\partial^\mu\varphi + iqA^\mu\varphi) \\ &= \partial_\mu\varphi^*\partial^\mu\varphi - iqA_\mu\varphi^*\partial^\mu\varphi + iqA^\mu\varphi\partial_\mu\varphi^* + q^2A_\mu A^\mu\varphi^*\varphi \\ &= \partial_\mu\varphi^*\partial^\mu\varphi + iq(\varphi\partial_\mu\varphi^* - \varphi^*\partial_\mu\varphi)A^\mu + q^2A_\mu A^\mu\varphi^*\varphi, \end{aligned} \tag{18}$$

and

$$\mathcal{L} = \partial_\mu\varphi^*\partial^\mu\varphi - V(\varphi^*\varphi) - j_\mu A^\mu + q^2A_\mu A^\mu\varphi^*\varphi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{19}$$

Now, since the functional form of the interacting matter Lagrangian is the same as the form of the free Lagrangian with $\mathcal{D}_\mu$ instead of $\partial_\mu$, the conserved Noether current in the presence of gauge interaction reads as [11]

$$\begin{aligned}J^\mu &= iq\varphi^*\frac{\partial\mathcal{L}_0}{\partial(\mathcal{D}_\mu\varphi)^*} + \varphi^* \to \varphi \\ &= iq\varphi^*\mathcal{D}^\mu\varphi + \varphi^* \to \varphi. \\ &= j^\mu - 2q^2\varphi^*\varphi A^\mu, \end{aligned} \tag{20}$$

the last two lines being valid for the KG case. The interaction term can now be written, up to second order terms [12] in $A_\mu$, as $\mathcal{L}_{\text{int}} = -J_\mu A^\mu$.

The equations of motion follow from Euler-Lagrange equations,

$$\frac{\delta S_{\text{tot}}}{\delta A_\mu} = \frac{\partial\mathcal{L}_{\text{tot}}}{\partial A_\mu} - \partial_\nu\frac{\partial\mathcal{L}_{\text{tot}}}{\partial(\partial_\nu A_\mu)} = 0, \tag{21}$$

which simply yield

$$\frac{\partial\mathcal{L}_{\text{int}}}{\partial A_\mu} = \partial_\nu\frac{\partial\mathcal{L}_A}{\partial(\partial_\nu A_\mu)}. \tag{22}$$

10. Remember that we call $\mathcal{L}_0 = F(\varphi, \varphi^*, \partial_\mu\varphi, \partial_\mu\varphi^*)$.

11. See e.g. V. Rubakov, Classical Theory of Gauge fields, Princeton University Press 2002, pp21-27.

12. These terms are particularly important, since they restore gauge invariance of the interaction Lagrangian which otherwise would not exhibit this gauge invariance property in the present form. Indeed, $J_\mu$ is gauge invariant, but $A_\mu$ is not.

The l.h.s. is $\frac{\partial \mathcal{L}_{\rm int}}{\partial A_\mu} = -j^\mu + 2q^2\varphi^*\varphi A^\mu = -J^\mu$ while at the r.h.s. we have $\frac{\partial \mathcal{L}_A}{\partial(\partial_\nu A_\mu)} = -\frac{1}{4}\frac{\partial}{\partial(\partial_\nu A_\mu)} F_{\alpha\beta}F^{\alpha\beta} = -\frac{1}{4}\left(\frac{\partial F_{\alpha\beta}}{\partial(\partial_\nu A_\mu)}F^{\alpha\beta} + F_{\alpha\beta}\frac{\partial F^{\alpha\beta}}{\partial(\partial_\nu A_\mu)}\right)$. Both terms in parenthesis are equal to $F^{\nu\mu} - F^{\mu\nu} = -2F^{\mu\nu}$ and eventually we obtain after multiplication by $-\frac{1}{4}$, [13]

$$\partial_\nu F^{\mu\nu} = -\partial_\nu F^{\nu\mu} = -J^\mu \tag{23}$$

and, since $F^{\mu\nu}$ is antisymmetric, we recover the conservation equation for $J^\mu$,

$$\partial_\mu J^\mu = 0. \tag{24}$$

# □ Non-Abelian (Yang-Mills) $SU(2)$ gauge theory

## □ Internal structure

The global phase transformation $\varphi(x) \to \exp(iq\alpha)\varphi(x)$ as mentioned above appears as an Abelian rotation in a two-dimensional (internal) space. This gauge transformation, when extended to local phase transformations $\alpha \to \alpha(x)$, generates the electromagnetic interaction. It is possible to generalize to non-Abelian gauge transformations by extending the internal (isospin) structure. The field is for example a 3-component real scalar field

$$\boldsymbol{\phi}(x) = \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \end{pmatrix} \tag{25}$$

and the transformation corresponds to a rotation in the internal space [14] (the corresponding charge is now written $g$)

$$\boldsymbol{\phi}(x) \to \exp(ig\boldsymbol{\tau\alpha})\boldsymbol{\phi}(x), \tag{26}$$

with $\boldsymbol{\tau}$ the generators (which do not commute, hence the name of non-Abelian) of the rotations in three dimensions

$$\tau^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \tau^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad \tau^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{27}$$

13. Note again that our choice of sign for the current density makes the expression coherent with the usual Maxwell equations.
14. See Weinberg II p3.

Use of bold font stands for vectors in the internal space and the scalar product (with $\cdot$ omitted) $\boldsymbol{\tau\alpha}$ is a $3 \times 3$ matrix,

$$\boldsymbol{\tau\alpha} = \begin{pmatrix} 0 & -i\alpha^3 & i\alpha^2 \\ i\alpha^3 & 0 & -i\alpha^1 \\ -i\alpha^2 & i\alpha^1 & 0 \end{pmatrix}. \tag{28}$$

The transformation (being a rotation in 3 dimensions) is non Abelian and it can be rewritten as [15] $\boldsymbol{\phi}(x) \to \boldsymbol{\phi}(x) - \boldsymbol{\alpha} \times \boldsymbol{\phi}(x)$. Here, $\boldsymbol{\alpha}$ is a vector in the internal space whose length $\alpha$ is the angle of rotation and whose direction is the rotation axis.

Rotations in three space dimensions are equivalent to $SU(2)$ transformations acting on complex two-component spinors [16]. The field is now represented by such a spinor,

$$\psi(x) = \begin{pmatrix} \varphi_\uparrow(x) \\ \varphi_\downarrow(x) \end{pmatrix}. \tag{29}$$

Each component is complex, but due to a normalization constraint we still have three independent real scalar fields. Under a rotation in the internal space, $\psi(x)$ changes into

$$\psi(x) \to \exp\left(\tfrac{1}{2} ig\boldsymbol{\sigma\alpha}\right) \psi(x) \tag{30}$$

where $\boldsymbol{\sigma}$ are the three Pauli matrices [17],

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{32}$$

which obey the Lie algebra

$$[\sigma^i, \sigma^j] = 2i\epsilon_{ijk}\sigma^k \tag{33}$$

15. See Ryder p108.
16. An $O(3)$ transformation on $\boldsymbol{\phi}$ corresponds to an $SU(2)$ transformation on $\psi = \begin{pmatrix} \varphi_\uparrow(x) \\ \varphi_\downarrow(x) \end{pmatrix}$ with $\phi^1 = \frac{1}{2}(\varphi_\downarrow^2 - \varphi_\uparrow^2)$, $\phi^2 = \frac{1}{2i}(\varphi_\uparrow^2 + \varphi_\downarrow^2)$, $\phi^3 = \varphi_\uparrow\varphi_\downarrow$, see Ryder pp32-38.
17. There is nothing very mysterious to introduce two component spinors and the Pauli matrices in the context of quantum mechanics. Indeed, the Pauli equation, which describes the non-relativistic spin-$\frac{1}{2}$ electron in an electromagnetic field reads as

$$\hat{H} \begin{pmatrix} \varphi_\uparrow(x) \\ \varphi_\downarrow(x) \end{pmatrix} = \left( \frac{1}{2m}(\vec{p} - q\vec{A})^2 - q\phi \ \mathbb{1} \begin{pmatrix} \varphi_\uparrow(x) \\ \varphi_\downarrow(x) \end{pmatrix} - \frac{q\hbar}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B} \begin{pmatrix} \varphi_\uparrow(x) \\ \varphi_\downarrow(x) \end{pmatrix} \right) = E \begin{pmatrix} \varphi_\uparrow(x) \\ \varphi_\downarrow(x) \end{pmatrix}. \tag{31}$$

Note that here the "hat" $\hat{H}$ notation stands for a 2 by 2 matrix.

with $\epsilon_{ijk}$ the totally antisymmetric tensor, and $\boldsymbol{\sigma\alpha}$ is a $2 \times 2$ matrix

$$\boldsymbol{\sigma\alpha} = \left(\begin{matrix} \alpha^3 & \alpha^1 - i\alpha^2 \\ \alpha^1 + i\alpha^2 & -\alpha^3 \end{matrix}\right. \tag{34}$$

Here $\frac{1}{2}\sigma^i$ are the generators of $SU(2)$ transformations.

The two representations can be written in a unified way in component form[18]. The (infinitesimal) transformation of the field (let say $\Psi(x)$, for $\boldsymbol{\phi}$ or $\psi$) is written

$$\delta\Psi_l(x) = ig\alpha^a(t^a)_l{}^m\Psi_m(x). \tag{35}$$

The $\alpha^a$'s are now the parameters of the *infinitesim* $l$ transformation. The superscript $a$ is used as internal space index, $l$ and $m$ denote the 2 components (resp. 3) in the $SU(2)$ representation (resp $SO(3)$) and $\boldsymbol{t}$ stands for the generator ($\boldsymbol{\sigma}$ (resp. $\boldsymbol{\tau}$)).

Together with the internal degrees of freedom, the fields of course depend on space-time position $x^\mu$.

## ☐ Pure matter field and Noether current

From now on, we choose the $SU(2)$ representation. Let $\mathcal{L}_0$ be a gauge invariant matter Lagrangian density

$$\mathcal{L}_0 = \partial_\mu\psi^\dagger\partial^\mu\psi - V(\psi^\dagger\psi) \tag{36}$$

where $\psi^\dagger(x) = (\varphi^*_\uparrow(x), \varphi^*_\downarrow(x))$ and $V(\psi^\dagger\psi)$ is a potential to be defined later.

The variation of the Lagrangian density yields

$$\delta\mathcal{L}_0 = \delta\psi^\dagger\frac{\partial\mathcal{L}_0}{\partial\psi^\dagger} + \delta(\partial_\mu\psi^\dagger)\frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\psi^\dagger)} + \frac{\partial\mathcal{L}_0}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\psi)}\delta(\partial_\mu\psi). \tag{37}$$

With the infinitesimal transformation

$$\delta\psi(x) = \tfrac{1}{2}ig\boldsymbol{\sigma\alpha}\psi(x), \tag{38}$$

$$\delta\psi^\dagger(x) = -\tfrac{1}{2}ig\psi^\dagger(x)\boldsymbol{\sigma\alpha}, \tag{39}$$

which can be written in matrix form,

$$\left(\begin{matrix}\delta\varphi_\uparrow(x) \\ \delta\varphi_\downarrow(x)\end{matrix}\right. = \tfrac{1}{2}ig\left(\begin{matrix} \alpha^3 & \alpha^1 - i\alpha^2 \\ \alpha^1 + i\alpha^2 & -\alpha^3 \end{matrix}\right. \left(\begin{matrix}\varphi_\uparrow(x) \\ \varphi_\downarrow(x)\end{matrix}\right. \tag{40}$$

$$(\delta\varphi^*_\uparrow(x), \delta\varphi^*_\downarrow(x)) = -\tfrac{1}{2}ig(\varphi^*_\uparrow(x), \varphi^*_\downarrow(x))\left(\begin{matrix} \alpha^3 & \alpha^1 + i\alpha^2 \\ \alpha^1 - i\alpha^2 & -\alpha^3 \end{matrix}\right. , \tag{41}$$

18. See Weinberg II p2.

and using the equations of motion

$$\frac{\partial \mathcal{L}_0}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \psi)} = 0, \qquad \frac{\partial \mathcal{L}_0}{\partial \psi^\dagger} - \partial_\mu \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \psi^\dagger)} = 0 \tag{42}$$

we obtain the variation of the Lagrangian density

$$\begin{aligned} \delta\mathcal{L}_0 &= \big(-\tfrac{1}{2} i g \psi^\dagger \boldsymbol{\sigma}\boldsymbol{\alpha}\big)\, \partial_\mu \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \psi^\dagger)} + \big(-\tfrac{1}{2} i g \partial_\mu \psi^\dagger \boldsymbol{\sigma}\boldsymbol{\alpha}\big) \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \psi^\dagger)} + (\psi^\dagger \to \psi) \\ &= -\boldsymbol{\alpha}\partial_\mu \boldsymbol{j}^\mu \end{aligned} \tag{43}$$

where the Noether current

$$\boldsymbol{j}^\mu = -\big(-\tfrac{1}{2} i g \psi^\dagger \boldsymbol{\sigma}\big) \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \psi^\dagger)} - \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \psi)} \big(\tfrac{1}{2} i g \boldsymbol{\sigma}\psi\big) \tag{44}$$

is conserved, $\partial_\mu \boldsymbol{j}^\mu = 0$, since the variation of the Lagrangian density vanishes for a symmetry.

The current density is a vector in internal space (isospin current density) [19]. With the Lagrangian density given in eqn. (**??**), the conserved current is explicitly given by

$$\boldsymbol{j}^\mu = \tfrac{1}{2} i g \psi^\dagger \boldsymbol{\sigma} \partial^\mu \psi - \partial^\mu \psi^\dagger \tfrac{1}{2} i g \boldsymbol{\sigma} \psi \tag{45}$$

$$\begin{aligned} &= \tfrac{1}{2} i g (\varphi_\uparrow^*, \varphi_\downarrow^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial^\mu \begin{pmatrix} \varphi_\uparrow \\ \varphi_\downarrow \end{pmatrix} \mathbf{i} - ()^* \to () \\ &\qquad + \tfrac{1}{2} i g (\varphi_\uparrow^*, \varphi_\downarrow^*) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \partial^\mu \begin{pmatrix} \varphi_\uparrow \\ \varphi_\downarrow \end{pmatrix} \mathbf{j} - ()^* \to () \\ &\qquad\qquad + \tfrac{1}{2} i g (\varphi_\uparrow^*, \varphi_\downarrow^*) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial^\mu \begin{pmatrix} \varphi_\uparrow \\ \varphi_\downarrow \end{pmatrix} \mathbf{k} - ()^* \to () \\ &= \tfrac{1}{2} i g (\varphi_\uparrow^* \partial^\mu \varphi_\downarrow + \varphi_\downarrow^* \partial^\mu \varphi_\uparrow)\mathbf{i} - ()^* \to () \\ &\qquad + \tfrac{1}{2} i g (-i\varphi_\uparrow^* \partial^\mu \varphi_\downarrow + i\varphi_\downarrow^* \partial^\mu \varphi_\uparrow)\mathbf{j} - ()^* \to () \\ &\qquad\qquad + \tfrac{1}{2} i g (\varphi_\uparrow^* \partial^\mu \varphi_\uparrow + \varphi_\downarrow^* \partial^\mu \varphi_\downarrow)\mathbf{k} - ()^* \to () \end{aligned} \tag{46}$$

where **i**, **j** et **k** are the unit vectors in isospin space.

19. See Quigg p34

### □ Introduction of a gauge potential [20]

Now we extend the formalism to local gauge transformations

$$\psi'(x) = \exp\left(\tfrac{1}{2}ig\boldsymbol{\sigma\alpha}(x)\right)\psi(x) \equiv \hat{G}(x)\psi(x), \tag{47}$$

(where $\hat{G}(x) = \exp\left(\tfrac{1}{2}ig\boldsymbol{\sigma\alpha}(x)\right)$ is a $2\times 2$ matrix) or locally to

$$\delta\psi(x) = \tfrac{1}{2}ig\boldsymbol{\sigma\alpha}(x)\psi(x), \tag{48}$$

but a problem occurs which will make the Lagrangian density $\mathcal{L}_0$ not gauge invariant. The fact that $\partial_\mu\psi$ does not obey the same gauge transformation than $\psi$ itself corrupts the transformation of the kinetic energy. We have $\partial_\mu\psi' = (\partial_\mu\hat{G})\psi + \hat{G}(\partial_\mu\psi)$. Let us introduce a covariant derivative [21]

$$\mathcal{D}_\mu \equiv \partial_\mu + ig\hat{B}_\mu. \tag{49}$$

Here as before we use the short notation $\partial_\mu$ for $\partial_\mu\hat{\mathbb{1}}$ with $\hat{\mathbb{1}}$ the 2 by 2 identity matrix. The context suffices to distinguish between $\partial_\mu$ and $\partial_\mu\hat{\mathbb{1}}$. $\hat{B}_\mu = \frac{1}{2}\boldsymbol{\sigma B}_\mu = \frac{1}{2}\sigma^a B^a_\mu$ (summation over $a$ understood) is another 2 by 2 matrix (in fact there is one such matrix for each of the 4 space-time components, $\boldsymbol{B}_\mu$ is a gauge potential (with three internal components which all are 4−space-time vectors)). We demand the following transformation

$$\mathcal{D}_\mu\psi(x) \to \mathcal{D}'_\mu\psi'(x) = \hat{G}(x)(\mathcal{D}_\mu\psi(x)). \tag{50}$$

We obtain $\mathcal{D}'_\mu\psi' = (\partial_\mu + ig\hat{B}'_\mu)\psi' = (\partial_\mu\hat{G})\psi + \hat{G}(\partial_\mu\psi) + ig\hat{B}'_\mu\hat{G}\psi$. From the requirement (**??**), this quantity should be equal to $\hat{G}(\partial_\mu + ig\hat{B}_\mu)\psi = \hat{G}(\partial_\mu\psi) + ig\hat{G}(\hat{B}_\mu\psi)$. It follows an equation for the transformation of $\hat{B}_\mu$, $ig\hat{B}'_\mu\hat{G}\psi = ig\hat{G}(\hat{B}_\mu\psi) - (\partial_\mu\hat{G})\psi$. Written in terms of operators, this equation is $\hat{B}'_\mu\hat{G} = \hat{G}\hat{B}_\mu + \frac{i}{g}\partial_\mu\hat{G}$. We multiply both sides by $\hat{G}^{-1}$ on the right to get

$$\hat{B}'_\mu = \hat{G}\hat{B}_\mu\hat{G}^{-1} + \frac{i}{g}(\partial_\mu\hat{G})\hat{G}^{-1} = \hat{G}\left(\hat{B}_\mu + \frac{i}{g}\hat{G}^{-1}(\partial_\mu\hat{G})\right)\hat{G}^{-1}. \tag{51}$$

In the case of electromagnetism, the local gauge transformation is performed by the operator (now an ordinary function) $G_{EM}(x) = \exp(iq\alpha(x))$ with

20. See Quigg pp55-57

21. In analogy with electromagnetism where the covariant derivative $i\hbar\mathcal{D}_\mu$ is given by $p_\mu - qA_\mu$ with $p_\mu = i\hbar\partial_\mu$, which yields $\mathcal{D}_\mu = \partial_\mu + i\frac{q}{\hbar}A_\mu$.

$\alpha(x)$ some function, and eqn. (**??**) leads to the known transformation of the gauge potential of electromagnetism,

$$A'_\mu = G_{EM} A_\mu G_{EM}^{-1} + \frac{i}{q}(\partial_\mu G_{EM}) G_{EM}^{-1} = A_\mu - \partial_\mu \alpha. \tag{52}$$

From eqn. (**??**) and the transformation of $\hat{B}_\mu = \frac{1}{2}\boldsymbol{\sigma}\boldsymbol{B}_\mu$, we can deduce the gauge transformation of $\boldsymbol{B}_\mu$ as well. Consider an infinitesimal gauge transformation

$$\hat{G}(x) = \hat{\mathbb{1}} + \tfrac{1}{2} i g \boldsymbol{\sigma}\boldsymbol{\alpha}(x). \tag{53}$$

Eqn. (**??**) reads as (to linear order in $\alpha^i$)

$$\tfrac{1}{2}\boldsymbol{\sigma}\boldsymbol{B}'_\mu = \tfrac{1}{2}\boldsymbol{\sigma}\boldsymbol{B}_\mu + \tfrac{1}{4} i g ((\boldsymbol{\sigma}\boldsymbol{\alpha})(\boldsymbol{\sigma}\boldsymbol{B}_\mu) - (\boldsymbol{\sigma}\boldsymbol{B}_\mu)(\boldsymbol{\sigma}\boldsymbol{\alpha})) - \tfrac{1}{2}\partial_\mu(\boldsymbol{\sigma}\boldsymbol{\alpha}). \tag{54}$$

The term in the middle, written in components, has the form

$$\tfrac{1}{2} i \alpha^j B_\mu^k (\sigma^j \sigma^k - \sigma^k \sigma^j) = \tfrac{1}{2} i \alpha^j B_\mu^k [\sigma^j, \sigma^k] = -\varepsilon_{jkl}(\alpha^j B_\mu^k)\sigma^l = -(\boldsymbol{\alpha} \times \boldsymbol{B}_\mu)\cdot\boldsymbol{\sigma} \tag{55}$$

and it follows that

$$\tfrac{1}{2}\boldsymbol{\sigma}\boldsymbol{B}'_\mu = \tfrac{1}{2}\boldsymbol{\sigma}\boldsymbol{B}_\mu - \tfrac{1}{2} g (\boldsymbol{\alpha} \times \boldsymbol{B}_\mu)\cdot\boldsymbol{\sigma} - \tfrac{1}{2}\partial_\mu(\boldsymbol{\sigma}\boldsymbol{\alpha}). \tag{56}$$

Another common expression uses the identity $(\boldsymbol{\alpha}\times\boldsymbol{B}_\mu)\cdot\boldsymbol{\sigma} = -2i\left[\frac{1}{2}\boldsymbol{\sigma}\boldsymbol{\alpha}, \frac{1}{2}\boldsymbol{\sigma}\boldsymbol{B}_\mu\right]$ such that

$$\tfrac{1}{2}\boldsymbol{\sigma}\boldsymbol{B}'_\mu = \tfrac{1}{2}\boldsymbol{\sigma}\boldsymbol{B}_\mu + i g \left[\tfrac{1}{2}\boldsymbol{\sigma}\boldsymbol{\alpha}, \tfrac{1}{2}\boldsymbol{\sigma}\boldsymbol{B}_\mu\right] - \tfrac{1}{2}\partial_\mu(\boldsymbol{\sigma}\boldsymbol{\alpha}). \tag{57}$$

We can also write directly

$$\boldsymbol{B}'_\mu = \boldsymbol{B}_\mu - g\boldsymbol{\alpha} \times \boldsymbol{B}_\mu - \partial_\mu \boldsymbol{\alpha}. \tag{58}$$

The gauge transformation of $\boldsymbol{B}_\mu$ appears as a gradient term (like in electromagnetism) plus a rotation in internal space.

### □ The field-strength tensor and field equations [22]

Let us introduce a field-strength tensor by the 2 by 2 matrix

$$\hat{F}_{\mu\nu} = \tfrac{1}{2}\boldsymbol{\sigma}\boldsymbol{F}_{\mu\nu} \tag{59}$$

22. See Quigg pp58-59

from which we construct the gauge-invariant kinetic energy

$$\mathcal{L}_{\text{field}} = -\tfrac{1}{4}\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu} = -\tfrac{1}{2}\,\mathrm{Tr}(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}), \tag{60}$$

where we used $\mathrm{Tr}(\sigma^a\sigma^b) = 2\delta^{ab}$[23]. The field-strength tensor is an observable quantity. It is thus supposed to be a "gauge scalar", that is to be independent of the choice of gauge[24] :

$$\hat{F}'_{\mu\nu} = \hat{G}\hat{F}_{\mu\nu}\hat{G}^{-1}. \tag{61}$$

A simple transcription of the QED Faraday tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is not satisfactory, but if we note that the QED Faraday tensor can also be written as

$$F_{\mu\nu} = \frac{1}{iq}[\mathcal{D}_\mu, \mathcal{D}_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + iq[A_\mu, A_\nu] \tag{62}$$

(the commutator vanishes in the Abelian case), we can define

$$\hat{F}_{\mu\nu} = \frac{1}{ig}[\mathcal{D}_\mu, \mathcal{D}_\nu] = \partial_\mu \hat{B}_\nu - \partial_\nu \hat{B}_\mu + ig[\hat{B}_\mu, \hat{B}_\nu]. \tag{63}$$

It is easy to check that this definition has the correct gauge invariance property.

In terms of "isovectors", the same relation becomes

$$\begin{aligned} \tfrac{1}{2}\boldsymbol{\sigma}\boldsymbol{F}_{\mu\nu} &= \tfrac{1}{2}\boldsymbol{\sigma}\partial_\mu\boldsymbol{B}_\nu - \tfrac{1}{2}\boldsymbol{\sigma}\partial_\nu\boldsymbol{B}_\mu + ig\left[\tfrac{1}{2}\boldsymbol{\sigma}\boldsymbol{B}_\mu, \tfrac{1}{2}\boldsymbol{\sigma}\boldsymbol{B}_\nu\right] \\ \boldsymbol{F}_{\mu\nu} &= \partial_\mu\boldsymbol{B}_\nu - \partial_\nu\boldsymbol{B}_\mu - g\boldsymbol{B}_\mu \times \boldsymbol{B}_\nu, \end{aligned} \tag{64}$$

where we used $\left[\tfrac{1}{2}\boldsymbol{\sigma}\boldsymbol{B}_\mu, \tfrac{1}{2}\boldsymbol{\sigma}\boldsymbol{B}_\nu\right] = \tfrac{1}{2}i(\boldsymbol{B}_\mu \times \boldsymbol{B}_\nu)\cdot\boldsymbol{\sigma}$. The non-Abelian character of the theory is obvious in the definition of the field-strength tensor. From the field Lagrangian (**??**) and the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}_{\text{field}}}{\partial B^a_\mu} - \partial_\nu \left[\frac{\partial\mathcal{L}_{\text{field}}}{\partial(\partial_\nu B^a_\mu)}\right] = 0, \tag{65}$$

one can deduce the equations of motion (equivalent to the Maxwell equations in the absence of matter charge current) :

$$\partial^\mu\boldsymbol{F}_{\mu\nu} - g\boldsymbol{B}^\mu \times \boldsymbol{F}_{\mu\nu} = 0. \tag{66}$$

23. This kinetic energy has the same form as in QED, $\mathcal{L}_{U(1)\ \text{field}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.
24. In the same sense than a Lorentz scalar (i.e. a contraction) does not depend on the reference frame.

Note that the equations of motion are not as simple as the familiar Maxwell equations which are linear. The Yang-Mills equations of motion on the other hand are not linear, and this is due to the fact that the gauge field carries the charge associated to the interaction. Thus, even in the absence of matter, the derivative of the field tensor does not vanish.

In the massive case (which is not gauge invariant), a term

$$m^2 \boldsymbol{B}_\mu \boldsymbol{B}^\mu \tag{67}$$

is added to the Lagrangian density and the (Proca-like) equations of motion become

$$\partial^\mu \boldsymbol{F}_{\mu\nu} - g\boldsymbol{B}^\mu \times \boldsymbol{F}_{\mu\nu} = m^2 \boldsymbol{B}_\nu. \tag{68}$$

### □ Construction of a gauge-invariant interaction [25]

Since $\boldsymbol{\alpha}(x)$ depends on space-time, the variations $\delta(\partial_\mu\psi)$ (or $\delta(\partial_\mu\psi^\dagger)$) contain an extra term which contributes to $\delta\mathcal{L}_0$

$$\begin{aligned}
\delta\mathcal{L}_0 &= \Big(-\tfrac{1}{2}ig\psi^\dagger\boldsymbol{\sigma}\boldsymbol{\alpha}(x)\Big)\,\partial_\mu\,\frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\psi^\dagger)}\Big] \\
&\qquad + \Big(-\tfrac{1}{2}ig\partial_\mu\left[\psi^\dagger\boldsymbol{\sigma}\right]\boldsymbol{\alpha}(x) - \tfrac{1}{2}ig\partial_\mu\left[\boldsymbol{\alpha}(x)\right]\psi^\dagger\boldsymbol{\sigma}\Big)\,\frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\psi^\dagger)} + (\psi^\dagger\to\psi) \\
&= -\boldsymbol{\alpha}(x)(\partial_\mu\boldsymbol{j}^\mu) - (\partial_\mu\boldsymbol{\alpha}(x))\boldsymbol{j}^\mu
\end{aligned} \tag{69}$$

The first term vanishes thanks to Noether theorem, but the second term, $-(\partial_\mu\boldsymbol{\alpha}(x))\boldsymbol{j}^\mu$ persists, so $\mathcal{L}_0$ is not gauge invariant under local gauge transformations. In order to compensate this new term, we must add another contribution to the Lagrangian,

$$\mathcal{L}_1 = -\boldsymbol{j}^\mu\boldsymbol{B}_\mu \tag{70}$$

and demand that $\boldsymbol{B}_\mu$ obeys the gauge transformation (**??**). Now,

$$\delta\mathcal{L}_0 + \delta\mathcal{L}_1 = -\boldsymbol{\alpha}(\partial_\mu\boldsymbol{j}^\mu) - (\partial_\mu\boldsymbol{\alpha})\boldsymbol{j}^\mu - \boldsymbol{j}^\mu\delta\boldsymbol{B}_\mu - \delta\boldsymbol{j}^\mu\boldsymbol{B}_\mu, \tag{71}$$

25. This section may be omitted. It presents step by step the construction of a gauge invariant Lagrangian which is obtained faster by the minimal coupling requirement presented later. The approach used here follows the presentation of Ryder pp96-98 in the case of Abelian $U(1)$ gauge symmetry.

where only the first term vanishes identically. Performing the variation of $\boldsymbol{j}^\mu$ yields

$$
\begin{aligned}
-\delta\boldsymbol{j}^\mu &= -\tfrac{1}{2}ig\delta\psi^\dagger\boldsymbol{\sigma}\partial^\mu\psi - \tfrac{1}{2}ig\psi^\dagger\boldsymbol{\sigma}\delta(\partial^\mu\psi) + (\psi^\dagger\to\psi) \\
&= -\tfrac{1}{2}ig\left(-\tfrac{1}{2}ig\psi^\dagger\boldsymbol{\sigma}\boldsymbol{\alpha}\right)\boldsymbol{\sigma}\partial^\mu\psi \\
&\qquad -\tfrac{1}{2}ig\psi^\dagger\boldsymbol{\sigma}\left(\tfrac{1}{2}ig\boldsymbol{\sigma}\partial^\mu\boldsymbol{\alpha}\psi + \tfrac{1}{2}ig\boldsymbol{\sigma}\boldsymbol{\alpha}\partial^\mu\psi\right) + (\psi^\dagger\to\psi) \\
&= \tfrac{1}{4}g^2\psi^\dagger\left[\boldsymbol{\sigma},\boldsymbol{\sigma}\boldsymbol{\alpha}\right]\partial^\mu\psi + \tfrac{1}{4}g^2\partial^\mu\psi^\dagger\left[\boldsymbol{\sigma},\boldsymbol{\sigma}\boldsymbol{\alpha}\right]\psi + \tfrac{1}{4}g^2\psi^\dagger\left\{\boldsymbol{\sigma},\boldsymbol{\sigma}\partial^\mu\boldsymbol{\alpha}\right\}\psi \\
&= -\tfrac{1}{2}ig^2\psi^\dagger\boldsymbol{\sigma}\times\boldsymbol{\alpha}\partial^\mu\psi - \tfrac{1}{2}ig^2\partial^\mu\psi^\dagger\boldsymbol{\sigma}\times\boldsymbol{\alpha}\psi + \tfrac{1}{2}g^2\partial^\mu\boldsymbol{\alpha}\psi^\dagger\psi
\end{aligned}
\tag{72}
$$

We have used the identities $[\boldsymbol{\sigma},\boldsymbol{\sigma}\boldsymbol{\alpha}] = -2i\boldsymbol{\sigma}\times\boldsymbol{\alpha}$ and $\{\boldsymbol{\sigma},\boldsymbol{\sigma}\partial^\mu\boldsymbol{\alpha}\} = 2\partial^\mu\boldsymbol{\alpha}$ which are proven by the use of Pauli matrices properties $[\sigma^i,\sigma^j] = 2i\varepsilon_{ijk}\sigma^k$ and $\{\sigma^i,\sigma^j\} = 2\delta_{ij}\mathbb{1}$. The three remaining terms of eqn. (**??**) are equal to

$$
-(\partial_\mu\boldsymbol{\alpha})\boldsymbol{j}^\mu = -\tfrac{1}{2}ig\psi^\dagger\boldsymbol{\sigma}\partial_\mu\boldsymbol{\alpha}\partial^\mu\psi + (\psi^\dagger\to\psi), \tag{73}
$$

$$
-\boldsymbol{j}^\mu\delta\boldsymbol{B}_\mu = -g\boldsymbol{j}^\mu\cdot(\boldsymbol{\alpha}\times\boldsymbol{B}_\mu) - \boldsymbol{j}^\mu\partial_\mu\boldsymbol{\alpha}, \tag{74}
$$

$$
\begin{aligned}
-\delta\boldsymbol{j}^\mu\boldsymbol{B}_\mu &= -\tfrac{1}{2}ig^2\psi^\dagger(\boldsymbol{\sigma}\times\boldsymbol{\alpha})\cdot\boldsymbol{B}_\mu\partial^\mu\psi - \tfrac{1}{2}ig^2\partial^\mu\psi^\dagger(\boldsymbol{\sigma}\times\boldsymbol{\alpha})\cdot\boldsymbol{B}_\mu\psi \\
&\qquad +\tfrac{1}{2}g^2\partial^\mu\boldsymbol{\alpha}\boldsymbol{B}_\mu\psi^\dagger\psi \\
&= +g\boldsymbol{j}^\mu\cdot(\boldsymbol{\alpha}\times\boldsymbol{B}_\mu) + \tfrac{1}{2}g^2\partial^\mu\boldsymbol{\alpha}\boldsymbol{B}_\mu\psi^\dagger\psi,
\end{aligned}
\tag{75}
$$

where we have used the cyclic property $(\boldsymbol{\sigma}\times\boldsymbol{\alpha})\cdot\boldsymbol{B}_\mu = (\boldsymbol{\alpha}\times\boldsymbol{B}_\mu)\cdot\boldsymbol{\sigma}$. The sum eventually gives only

$$
\delta\mathcal{L}_0 + \delta\mathcal{L}_1 = \tfrac{1}{2}g^2\partial^\mu\boldsymbol{\alpha}\boldsymbol{B}_\mu\psi^\dagger\psi \tag{76}
$$

We still have to add another term which should eventually make the whole Lagrangian gauge invariant,

$$
\mathcal{L}_2 = \tfrac{1}{4}g^2\psi^\dagger\boldsymbol{B}_\mu\boldsymbol{B}^\mu\psi = \tfrac{1}{4}g^2\boldsymbol{B}_\mu\boldsymbol{B}^\mu\psi^\dagger\psi. \tag{77}
$$

The variation of $\mathcal{L}_2$ reads as

$$
\begin{aligned}
\delta\mathcal{L}_2 &= \tfrac{1}{4}g^2(\boldsymbol{B}_\mu\delta\boldsymbol{B}^\mu\psi^\dagger\psi + \delta\boldsymbol{B}_\mu\boldsymbol{B}^\mu\psi^\dagger\psi + \boldsymbol{B}_\mu\boldsymbol{B}^\mu\delta(\psi^\dagger\psi)) \\
&= \tfrac{1}{2}g^2\boldsymbol{B}_\mu\delta\boldsymbol{B}^\mu\psi^\dagger\psi \\
&= -\tfrac{1}{2}g^2\boldsymbol{B}_\mu(g(\boldsymbol{\alpha}\times\boldsymbol{B}^\mu) + \partial^\mu\boldsymbol{\alpha})\psi^\dagger\psi \\
&= -\tfrac{1}{2}g^2\boldsymbol{B}_\mu\partial^\mu\boldsymbol{\alpha}\psi^\dagger\psi
\end{aligned}
\tag{78}
$$

and we obtain the expected vanishing variation

$$
\delta\mathcal{L}_0 + \delta\mathcal{L}_1 + \delta\mathcal{L}_2 = 0 \tag{79}
$$

which proves that the gauge invariant interaction in the presence of gauge fields contains two terms,

$$\mathcal{L}_1 + \mathcal{L}_2 = -\boldsymbol{j}^\mu \boldsymbol{B}_\mu + \tfrac{1}{4} g^2 \boldsymbol{B}_\mu \boldsymbol{B}^\mu \psi^\dagger \psi. \tag{80}$$

The kinetic energy of the gauge field itself was not included, like the free particle contribution of eqn (**??**).

## □ Covariant derivative, minimal coupling

The introduction of the gauge covariant derivative facilitates the calculations. The action should not depend on the gauge choice, since the equations of motion are independent of the gauge. The Lagrangian density should thus be a gauge scalar. The potential term is already a gauge scalar, since it depends only on $\psi^\dagger\psi$ which transforms covariantly according to $\psi'^\dagger\psi' = (\psi^\dagger \hat{G}^{-1})(\hat{G}\psi)$. In order to become manifestly gauge covariant, the kinetic term should be written as $(\mathcal{D}_\mu\psi)^\dagger(\mathcal{D}^\mu\psi)$, since the covariant derivative of the field $\mathcal{D}_\mu\psi$ was constructed for the purpose of obeying the same gauge transformation than the field itself, $(\mathcal{D}_\mu{}'\psi')^\dagger(\mathcal{D}^{\mu\prime}\psi') = (\mathcal{D}_\mu\psi)^\dagger\hat{G}^{-1}\hat{G}(\mathcal{D}^\mu\psi)$.

The minimal coupling is the prescription that the interaction with the gauge field is obtained by the replacement $\partial_\mu\psi \to \mathcal{D}_\mu\psi$ is the Lagrangian density of eqn. (**??**) [26] :

$$\begin{aligned}
\mathcal{L} &= (\mathcal{D}_\mu\psi)^\dagger(\mathcal{D}^\mu\psi) - V(\psi^\dagger\psi) & (81)\\
&= \left(\partial_\mu - \tfrac{1}{2} ig\boldsymbol{\sigma}\boldsymbol{B}_\mu\right)\psi^\dagger \left(\partial^\mu + \tfrac{1}{2} ig\boldsymbol{\sigma}\boldsymbol{B}^\mu\right)\psi - V(\psi^\dagger\psi) \\
&= \partial_\mu\psi^\dagger\partial^\mu\psi - \tfrac{1}{2} ig\psi^\dagger\boldsymbol{\sigma}\boldsymbol{B}_\mu\partial^\mu\psi + \tfrac{1}{2} ig\partial^\mu\psi^\dagger\boldsymbol{\sigma}\boldsymbol{B}^\mu\psi \\
&\qquad + \tfrac{1}{4} g^2\psi^\dagger(\boldsymbol{\sigma}\boldsymbol{B}_\mu)(\boldsymbol{\sigma}\boldsymbol{B}^\mu)\psi - V(\psi^\dagger\psi) \\
&= \partial_\mu\psi^\dagger\partial^\mu\psi - \boldsymbol{j}^\mu\boldsymbol{B}_\mu + \tfrac{1}{4} g^2\boldsymbol{B}_\mu\boldsymbol{B}^\mu\psi^\dagger\psi - V(\psi^\dagger\psi) & (82)
\end{aligned}$$

where in the last term use has been made of the identity

$$(\boldsymbol{\sigma}\boldsymbol{B}_\mu)(\boldsymbol{\sigma}\boldsymbol{B}^\mu) = \begin{pmatrix} B_1B^1 + B_2B^2 + B_3B^3 & 0 \\ 0 & B_1B^1 + B_2B^2 + B_3B^3 \end{pmatrix} = \boldsymbol{B}_\mu\boldsymbol{B}^\mu\mathbb{1}. \tag{83}$$

This operation is called gauging the Lagrangian. Note that in the case of fermionic particles, we have to use the Dirac Lagrangian density $i\bar{\psi}\gamma^\mu\partial_\mu\psi$,

26. In an expression such that $\left(\partial_\mu - \frac{1}{2} ig\boldsymbol{\sigma}\boldsymbol{B}_\mu\right)\psi^\dagger$, it is understood that $\boldsymbol{\sigma}\boldsymbol{B}_\mu$ acts on $\psi^\dagger$ *on the left*.

with $\bar{\psi} = \psi^\dagger \gamma^0$ the adjoint spinor and $\gamma^\mu$ the Dirac matrices[27] instead of the kinetic energy $\partial_\mu \psi^\dagger \partial^\mu \psi$, and the gauge invariant Lagrangian becomes

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \mathcal{D}_\mu - m)\psi - \tfrac{1}{4}\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}. \tag{84}$$

## □ Conserved current in the presence of gauge fields

In the presence of gauge fields, the conserved Noether current can be written in terms of the covariant derivative,

$$\boldsymbol{J}^\mu = -\left(-\tfrac{1}{2}ig\psi^\dagger\boldsymbol{\sigma}\right)\frac{\partial\mathcal{L}_0}{\partial((\mathcal{D}_\mu\psi)^\dagger)} - \frac{\partial\mathcal{L}_0}{\partial(\mathcal{D}_\mu\psi)}\left(\tfrac{1}{2}ig\boldsymbol{\sigma}\psi\right) \tag{85}$$

In the case of the Lagrangian (**??**), it becomes

$$\begin{aligned}\boldsymbol{J}^\mu &= \tfrac{1}{2}ig\psi^\dagger\boldsymbol{\sigma}\mathcal{D}^\mu\psi - (\mathcal{D}^\mu\psi)^\dagger\tfrac{1}{2}ig\boldsymbol{\sigma}\psi \\ &= \tfrac{1}{2}ig\psi^\dagger\boldsymbol{\sigma}\left(\partial^\mu\psi + \tfrac{1}{2}ig\boldsymbol{\sigma}\boldsymbol{B}^\mu\psi\right) - \left(\partial^\mu\psi^\dagger - \tfrac{1}{2}ig\boldsymbol{\sigma}\boldsymbol{B}^\mu\psi^\dagger\right)\tfrac{1}{2}ig\boldsymbol{\sigma}\psi \\ &= \boldsymbol{j}^\mu - \tfrac{1}{4}g^2\psi^\dagger\left[\boldsymbol{\sigma},\boldsymbol{\sigma}\boldsymbol{B}^\mu\right]\psi.\end{aligned} \tag{86}$$

We see that the conserved current in the presence of gauge fields has two contributions, one coming from the ordinary matter current and the other from the gauge field itself. Using the identity $[\boldsymbol{\sigma},\boldsymbol{\sigma}\boldsymbol{B}^\mu] = -2i\boldsymbol{\sigma}\times\boldsymbol{B}^\mu$, we get

$$\boldsymbol{J}^\mu = \boldsymbol{j}^\mu + \tfrac{1}{2}ig^2\psi^\dagger\boldsymbol{\sigma}\times\boldsymbol{B}^\mu\psi. \tag{87}$$

In component form we have

$$J^{a\mu} = j^{a\mu} - \frac{1}{2}ig^2\psi^\dagger\varepsilon_{abc}\sigma^b B^{c\mu}\psi. \tag{88}$$

The current density $\boldsymbol{J}^\mu$ is conserved in the ordinary sense[28],

$$\partial_\mu\boldsymbol{J}^\mu = 0 \tag{89}$$

while $\boldsymbol{j}^\mu$ satisfies a gauge-covariant conservation law

$$\mathcal{D}_\mu\boldsymbol{j}^\mu = 0. \tag{90}$$

27. See e.g. Ryder pp43-46
28. See Weinberg II pp12-13

Now, the Euler-Lagrange equations of the matter field in the presence of the gauge field contain a new term,

$$\frac{\partial \mathcal{L}_{\text{int}}}{\partial B^a_\mu} + \frac{\partial \mathcal{L}_{\text{field}}}{\partial B^a_\mu} \left[ - \partial_\nu \frac{\partial \mathcal{L}_{\text{field}}}{\partial(\partial_\nu B^a_\mu)} \right] = 0, \tag{91}$$

and lead to the equations of motion in the presence of charged matter :

$$\partial^\mu \boldsymbol{F}_{\mu\nu} - g\boldsymbol{B}^\mu \times \boldsymbol{F}_{\mu\nu} = \boldsymbol{J}_\nu. \tag{92}$$

# □ Spontaneous gauge symmetry breaking

## □ Spontaneous breaking of global symmetries

### □ Discrete symmetries

Let us first consider spontaneous breaking of a global *discrete* symmetry. It is illustrated by the case of the real scalar field,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi(x)), \tag{93}$$

where $V(\phi(x))$ is a potential which depends on the field configuration. We will consider two cases

$$V(\phi) = \frac{\lambda}{4}\phi^4 \pm \frac{\mu^2}{2}\phi^2, \quad \lambda, \mu^2 > 0. \tag{94}$$

Case 1 with the sign + corresponds to a scalar field theory with square mass $\mu^2$. Let us first build the Hamiltonian,

$$\mathcal{H} = \tfrac{1}{2}(\partial_0\phi)^2 + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi). \tag{95}$$

The field with the lowest energy (also called the ground state configuration, which would be denoted $\phi_0 = \langle\Omega|\hat{\phi}|\Omega\rangle$ in the quantized version of the theory) is a constant field which minimizes the potential. In case 1, it corresponds to a vanishing field $\phi_0 = 0$. The discrete $Z_2$ symmetry $\phi \to -\phi$ of the Lagrangian is also a symmetry of the ground state. In case 2, with sign − in the potential, the constant ground state field is given by

$$\phi_0 = \pm\frac{\mu}{\sqrt{\lambda}} = \pm v. \tag{96}$$

While the Lagrangian still possesses the $\phi \to -\phi$ symmetry, in any of the two degenerate ground states, this $Z_2$ symmetry is broken. This situation occurs in the low temperature phase of second order phase transitions, when an ordered ground state emerges (for example a ferromagnetic ground state), which does not respect the full symmetry of the Hamiltonian (e.g. the "up-down" symmetry in a Ising model, or rotational symmetry (this is a continuous symmetry in this case) in the case of an Heisenberg model). It is instructive to study the field fluctuations around the ground state. For this purpose, we let

$$\phi = v + h, \quad h \ll v, \tag{97}$$

($v$ is chosen positive without loss of generality) and we remind that $V'(\phi) = \lambda\phi^3 - \mu^2\phi$, $V''(\phi) = 3\lambda\phi^2 - \mu^2$, $V'''(\phi) = 6\lambda\phi$, and $V''''(\phi) = 6\lambda$. The potential is now

$$\begin{aligned} V(\phi) &= V(v) + hV'(v) + \tfrac{1}{2}h^2V''(v) + \tfrac{1}{6}h^3V'''(v) + \tfrac{1}{24}h^4V''''(v) \\ &= -\tfrac{1}{4}\frac{\mu^4}{\lambda} + 0 + \tfrac{1}{2}2\mu^2h^2 + 2\sqrt{\lambda}\mu h^3 + \tfrac{1}{4}\lambda h^4. \end{aligned} \tag{98}$$

We note that the new field $v$ acquired a mass $\sqrt{2}\mu$ (the coefficient of the quadratic term in $h$ is now positive).

☐ **Continuous symmetries**

Spontaneous breaking of a global *continuous* symmetry can be encountered in the $SO(2)$ model (rotations in the plane). We consider now a theory with two real scalar fields $\phi_1(x)$ and $\phi_2(x)$, and with the potential

$$V(\phi_1, \phi_2) = \tfrac{1}{4}\lambda(\phi_1^2 + \phi_2^2 - v^2)^2 = \tfrac{1}{4}\lambda(|\boldsymbol{\phi}|^2 - v^2)^2. \tag{99}$$

The fields $\phi_1$ and $\phi_2$ are massless (the coefficients of the quadratic terms are negative) and the theory is invariant under rotations in the plane,

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{100}$$

The minima of the potential lie on the circle

$$|\boldsymbol{\phi}_0|^2 = \phi_{10}^2 + \phi_{20}^2 = v^2. \tag{101}$$

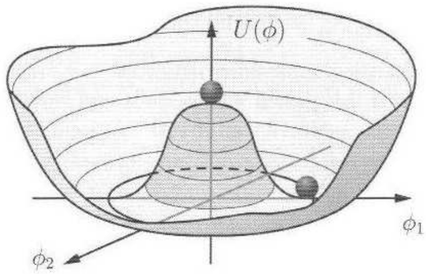


Figure 1 – The “Mexican hat potential” (from E.A. Paschos, Electroweak theory, Cambridge University Press, Cambridge 2007.

As a result of the continuous symmetry, they are infinitely degenerate. In order to analyze the field fluctuations around the minimum, we choose a particular vacuum state $\phi_{10} = v$, $\phi_{20} = 0$ and denote the fluctuations by

$$\phi_1 = v + h_1, \quad \phi_2 = h_2 \tag{102}$$

in terms of which the potential becomes

$$V(h_1, h_2) = \tfrac{1}{4}\lambda(h_1^2 + h_2^2 + 2vh_1)^2. \tag{103}$$

Expansion of this potential shows that $h_1$ becomes massive while $h_2$ remains massless, and the appearance of cubic terms breaks the original $SO(2)$ symmetry. The massless field is called a Goldstone mode (or Nambu-Goldstone mode). It is easy to understand why $h_2$ remains massless while $h_1$ acquired a mass : close to the minimum which we have selected, $\phi_1$ fluctuations have to survive to the potential growth, these are amplitude fluctuations in a polar representation of the model, while $\phi_2$ fluctuations correspond to phase fluctuations which do not cost any energy.

### □ Spontaneous breaking of local symmetries

A new phenomenon occurs with local gauge theories, where the selection of a particular minimum and the fluctuations around this minimum lead to massive gauge fields which would otherwise be forbidden, since mass terms for the gauge field would break gauge invariance. At the same time, the Goldstone mode disappears.

We consider the Lagrangian density of $U(1)$ gauge theory,

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\mathcal{D}_\mu\phi)^*(\mathcal{D}^\mu\phi) - V(\phi^*\phi), \tag{104}$$

with the potential

$$V(\phi^*\phi) = -\mu^2\phi^*\phi + \lambda(\phi^*\phi)^2. \tag{105}$$

As we have seen before, the theory is invariant under the gauge transformation corresponding to a local rotation of the scalar field in the complex plane

$$\begin{aligned} \phi(x) &\rightarrow e^{iq\alpha(x)}\phi(x) && (106)\\ A_\mu(x) &\rightarrow A_\mu(x) - \partial_\mu\alpha(x). && (107) \end{aligned}$$

Let us define two real fields $\theta(x)$ and $h(x)$, associated to the phase and the amplitude fluctuations around a particular (chosen real) minimum $v = (\mu^2/2\lambda)^{1/2}$,

$$\phi(x) = e^{i\theta(x)/v}\frac{1}{\sqrt{2}}(v + h(x)). \tag{108}$$

The local gauge transformation defined by $q\alpha(x) = -\theta(x)/v$ eliminates $\theta(x)$, since

$$\begin{aligned} \phi'(x) &= e^{-i\theta(x)/v}\phi(x) = \frac{1}{\sqrt{2}}(v + h(x)), && (109)\\ A'_\mu(x) &= A_\mu(x) + \frac{1}{qv}\partial_\mu\theta(x). && (110) \end{aligned}$$

The net effect in the Lagrangian density is the following,

$$\mathcal{L} = -\tfrac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + (\mathcal{D}'_\mu\phi')^*(\mathcal{D}'^\mu\phi') + \tfrac{1}{2}\mu^2(v + h^2(x))^2 - \tfrac{1}{4}\lambda(v + h(x))^4, \tag{111}$$

with $\mathcal{D}'_\mu = \partial_\mu + iqA'_\mu$. The kinetic energy term generates the mass for the gauge field $A'_\mu$ :

$$(\mathcal{D}'_\mu\phi')^*(\mathcal{D}'^\mu\phi') = \tfrac{1}{2}\partial_\mu h\partial^\mu h + \tfrac{1}{2}q^2A'_\mu A'^\mu(v^2 + 2hv + h^2), \tag{112}$$

and, as we announced, the Goldstone mode $\theta(x)$ was absorbed in the redefinition of the gauge field.

This mechanism is known in condensed matter physics as the Anderson mechanism (see next section), and it occurs in superconductivity, where the non-zero mass (which also defines a characteristic length scale) of the gauge field is responsible for the Meissner effect (the fact that the magnetic field is expelled from the bulk of the material). In particle physics, this mechanism enables to give a mass to the gauge bosons, as we discuss below. This is known in this context as the Higgs mechanism.

## ☐ The Anderson mechanism [29]

We will first reproduce all the generic arguments given before in the relativistic $U(1)$ case before considering the application to superconductivity.

### ☐ Gauge invariance in non relativistic quantum mechanics

Since we are interested in this section by non relativistic quantum mechanics, we will not distinguish between contravariant and covariant indices, i.e. $x_i = x, y, z$ and $\partial_i = \frac{\partial}{\partial x_i}$. Summation is understood as soon as an index is repeated in an expression. The space part of an expression like $\partial_\mu \psi^* \partial^\mu \psi$ will thus simply be denoted as $\partial_i \psi^* \partial^i \psi = -\partial_i \psi^* \partial_i \psi$, and $\partial_i^2 \psi = \partial_i \partial_i \psi$ stands for $\vec{\nabla}^2 \psi$. Due to this non-covariant notation, there are other minus signs here and there, for instance in the definition of the field tensor $F_{ij} = -(\partial_i A_j - \partial_j A_i)$.

The Lagrangian density for non relativistic quantum mechanics is given by an expression due to Jordan and Wigner (here written in a symmetric form)

$$\mathcal{L}_0 = \tfrac{1}{2} i\hbar(\psi^* \dot{\psi} - \dot{\psi}^* \psi) - \frac{\hbar^2}{2m} \partial_i \psi^* \partial_i \psi - V \psi^* \psi. \tag{113}$$

The Euler-Lagrange equation (variation with respect to $\psi^*$) indeed leads the Schrödinger equation,

$$\begin{aligned} \frac{\partial \mathcal{L}_0}{\partial \psi^*} &= \tfrac{1}{2} i\hbar \dot{\psi} - V\psi, & (114) \\ \partial_t \left( \frac{\partial \mathcal{L}_0}{\partial \dot{\psi}^*} \right. &= -\tfrac{1}{2} i\hbar \dot{\psi}, & (115) \\ \partial_i \left( \frac{\partial \mathcal{L}_0}{\partial (\partial_i \psi^*)} \right. &= -\frac{\hbar^2}{2m} \partial_i^2 \psi. & (116) \end{aligned}$$

and we have

$$i\hbar \dot{\psi} = -\frac{\hbar^2}{2m} \partial_i^2 \psi + V\psi. \tag{117}$$

Once we have noticed that the Lagrangian density is a function of $\psi$, $\dot{\psi}$, and $\partial_i \psi$ (as well as their complex conjugates), its variation under an infinitesimal

29. Caution : in all this section we forget about the covariant notation, all indices are space indices written as subscripts and summed over when repeated. See the beginning of the next paragraph for more detailed explanations.

global gauge transformation $\delta\psi = \frac{i}{\hbar}e\alpha\psi$ leads to

$$\begin{aligned}
\delta\mathcal{L}_0 &= -\frac{i}{\hbar}e\alpha \Bigg[ \psi^*\partial_i\left(\frac{\partial\mathcal{L}_0}{\partial(\partial_i\psi^*)}\right. + \psi^*\partial_t\left(\frac{\partial\mathcal{L}_0}{\partial(\partial_i\psi^*)}\right. + \partial_i\psi^*\frac{\partial\mathcal{L}_0}{\partial(\partial_i\psi^*)} + \dot{\psi}^*\frac{\partial\mathcal{L}_0}{\partial\dot{\psi}^*} \\
&\qquad + (\psi^* \longleftrightarrow \psi)\Bigg] \\
&= -\alpha[\partial_i j_i + \partial_t\rho], \qquad (118)
\end{aligned}$$

(use has been made of the equations of motion) where

$$\begin{aligned}
j_i &= \frac{e\hbar}{2mi}\left(\psi^*(\partial_i\psi) - (\partial_i\psi^*)\psi\right), \qquad (119)\\
\rho &= e\psi^*\psi. \qquad (120)
\end{aligned}$$

Notice that this continuity equation is usually written in the standard form

$$\vec{\nabla}\vec{j} + \frac{\partial\rho}{\partial t} = 0. \qquad (121)$$

In the presence of an electromagnetic field (relativistic in essence), we use the minimal coupling

$$\mathcal{D}_i = \partial_i - \frac{i}{\hbar}eA_i \qquad (122)$$

and we add the field Lagrangian contribution (for further purpose, we will only consider the magnetic contribution and only in a static situation, i.e. $-\frac{1}{4}F_{ij}F_{ij}$) ($F_{ij} = -(\partial_i A_j - \partial_j A_i)$),

$$\mathcal{L} = \tfrac{1}{2}i\hbar(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - \frac{\hbar^2}{2m}(\mathcal{D}_i\psi)^*(\mathcal{D}_i\psi) - V\psi^*\psi - \tfrac{1}{4}F_{ij}F_{ij}. \qquad (123)$$

The gauge field $A_i$ is changed by a local gauge transformation,

$$\begin{aligned}
\psi'(x) &= \psi(x)e^{\frac{i}{\hbar}e\alpha(x)}, \qquad (124)\\
A_i'(x) &= A_i(x) - \partial_i\alpha(x), \qquad (125)
\end{aligned}$$

but the field tensor $F_{ij}$ is unaffected. The equations of motion in the presence of the gauge field are modified [30],

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial A_j} &= \frac{e\hbar}{2mi}[\psi^*(\partial_j\psi) - (\partial_j\psi^*)\psi] - \frac{e^2}{m}A_j\psi^*\psi, \qquad (126)\\
\partial_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_i A_j)}\right. &= -\frac{1}{4}\frac{\partial}{\partial(\partial_i A_j)}(F_{kl}F_{kl}) \\
&= \partial_i F_{ij}, \qquad (127)
\end{aligned}$$

30. Note here a modification in the usual signs.

leading to Maxwell equations

$$\partial_i F_{ij} = J_j, \tag{128}$$

where the current density is also recovered from the expression

$$\begin{aligned} J_i &= \frac{e\hbar}{2mi}\left(\psi^*(\mathcal{D}_i\psi) - (\mathcal{D}_i\psi)^*\psi\right) \\ &= \frac{e\hbar}{2mi}\left(\psi^*(\partial_i\psi) - (\partial_i\psi^*)\psi\right) - \frac{e^2}{m}A_i\psi^*\psi. \end{aligned} \tag{129}$$

The Schrödinger equation follows from the variation w.r.t. $\psi^*$,

$$\frac{\partial\mathcal{L}}{\partial\psi^*} = \tfrac{1}{2}i\hbar\dot{\psi} - V\psi - \frac{\hbar^2}{2m}\left(\frac{i}{\hbar}eA_i\partial_i\psi + \frac{e^2}{\hbar^2}A_iA_i\psi\right., \tag{130}$$

$$\partial_t\left(\frac{\partial\mathcal{L}_0}{\partial\dot{\psi}^*}\right. = -\tfrac{1}{2}i\hbar\dot{\psi}, \tag{131}$$

$$\partial_i\left(\frac{\partial\mathcal{L}_0}{\partial(\partial_i\psi^*)}\right. = -\frac{\hbar^2}{2m}\left(\partial_i^2\psi - \frac{i}{\hbar}eA_i\psi\right.. \tag{132}$$

Collecting the different terms, we get

$$i\hbar\dot{\psi} = V\psi + \frac{1}{2m}[-i\hbar\partial_i - eA_i]^2\psi \tag{133}$$

provided that the Coulomb gauge $\partial_i A_i = 0$ is chosen.

□ **Gauge symmetry breaking**

We will now suppose that the gauge symmetry is spontaneously broken, i.e. the uniform ground state wave function which minimizes the potential energy is allowed to amplitude and phase fluctuation (for convenience, the phase fluctuations are removed by a local gauge transformation) and the amplitude fluctuations couple to the gauge field in such a way that the gauge field becomes massive. The initial Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}i\hbar(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - \frac{\hbar^2}{2m}(\mathcal{D}_i\psi)^*(\mathcal{D}_i\psi) - V(\psi^*\psi) - \tfrac{1}{4}F_{ij}F_{ij} \tag{134}$$

is gauge invariant. The potential energy has a minimum $|\psi_0|$ (e.g. in the following calculations $V(\psi^*\psi) = -\mu^2\psi^*\psi + \lambda(\psi^*\psi)^2$ has a minimum at $|\psi_0| = \sqrt{\mu^2/2\lambda}$) which can be chosen real positive $\psi_0$. We now allow for local amplitude and phase fluctuations around this minimum,

$$\psi(x) = (\psi_0 + h(x))e^{i\theta(x)}, \tag{135}$$

with $h(x)$ and $\theta(x)$ two real functions (and $h(x)$ small w.r.t. $\psi_0$).

A convenient gauge transformation

$$\psi'(x) = \psi(x) e^{\frac{i}{\hbar} e\alpha(x)}, \qquad \frac{e}{\hbar}\alpha(x) = \theta(x) \tag{136}$$

makes the analysis simpler, since it eliminates the phase fluctuations,

$$\psi'(x) = \psi_0 + h(x) \tag{137}$$

and also changes the gauge field

$$A'_i(x) = A_i(x) - \frac{\hbar}{e}\partial_i \theta(x). \tag{138}$$

The Lagrangian density is left unchanged by the gauge transformation, but written in terms of the gauged variables it reads

$$\mathcal{L} = \tfrac{1}{2} i\hbar(\psi'^* \dot{\psi}' - \dot{\psi}'^* \psi') - \frac{\hbar^2}{2m}(\mathcal{D}'_i \psi')^* (\mathcal{D}'_i \psi') - V(\psi'^* \psi') - \tfrac{1}{4} F'_{ij} F'_{ij}. \tag{139}$$

The first term is identically zero, the second term once expanded leads to

$$-\frac{\hbar^2}{2m}\left(\partial_i h \partial_i h + \frac{e^2}{\hbar^2} A'_i A'_i (\psi_0 + h)^2 \right., \tag{140}$$

the third term to

$$-\mu^2(\psi_0 + h)^2 + \lambda(\psi_0 + h)^4 \tag{141}$$

(or any other form depending on the potential), and the last gauge invariant term

$$-\tfrac{1}{4} F'_{ij} F'_{ij} \tag{142}$$

has the usual form for the field kinetic energy, but here given in terms of the gauged vector potential. The essential novelty stands in the second term where an extra dependence of the gauge field occurs, $\sim \psi_0 A_i^2$, coupled to the ground state expectation value. This is a mass term which is also denoted as

$$\tfrac{1}{2} m_A^2 = \frac{e^2}{\hbar^2}\psi_0^2. \tag{143}$$

Variation of $\mathcal{L}$ w.r.t. $h(x)$ leads to the equation of motion known as the Ginzburg-Landau equation in the context of superconductivity,

$$\frac{\partial \mathcal{L}}{\partial h(x)} = \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i h(x))}, \tag{144}$$

$$\frac{e^2}{m} A_i'^2(x)(\psi_0 + h(x)) + 2\mu^2(\psi_0 + h(x)) - 4\lambda(\psi_0 + h(x))^2 = \frac{\hbar^2}{m}\partial_i^2 h(x) \tag{145}$$

This equation contains information about the Cooper pair wave function $\psi_0 + h(x)$ and it involves a characteristic length scale known as the coherence length,

$$\xi^{-2} = \frac{2m\mu^2}{\hbar^2}. \tag{146}$$

The coherence length $\xi$ is for example a measure of the length scale needed to attain the condensate wave function in the bulk of a superconductor from a free surface.

Variation of $\mathcal{L}$ w.r.t. the gauge field leads to

$$\frac{\partial \mathcal{L}}{\partial A_j'(x)} = \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i A_j'(x))}, \tag{147}$$

$$-\frac{e^2}{m} A_i'(x)(\psi_0 + h(x))^2 = \partial_i F_{ij}'. \tag{148}$$

The l.h.s. corresponds to the London current density $J_j$ (proportional to the gauge field instead of the usual Ohm law $J_j = \sigma E_j$ in a normal metal). This equation also involves a typical length scale $\kappa$ inversely proportional to the Cooper pair wave function,

$$\kappa^{-2} = \frac{e^2}{m}\psi_0^2. \tag{149}$$

This parameter gives an information about the length scale needed to expel the gauge field from the bulk of a superconductor, a phenomenon known as the Meissner effect. It is completely governed by the gauge field mass, $\kappa^{-2} = \frac{\hbar^2}{2m} m_A^2$.

The phenomenology of type I and type II superconductors is essentially described in terms of the two length scales $\xi$ and $\kappa$. If $\kappa \ll \xi$, the magnetic field does not penetrate at all in the bulk of a superconductor (type I), while in the other limit, there exist regions of normal phase with non zero magnetic field (Abrikosov vortices) inside superconducting regions (mixed phase of type II superconductors).

### □ $SU(2)_W \times U(1)_Y$

The electroweak symmetry breaking scenario [31] discovered by Salam and Weinberg describes the emergence of the present structure of electromagnetic and weak interactions as the broken gauge symmetry phase of a symmetric (unbroken) phase $SU(2)_W \times U(1)_Y$ which existed in earlier times (higher energy scales) of the Universe. With the spontaneous symmetry breaking scenario, some of the bosonic degrees of freedom (the gauge fields) acquire mass. In the symmetric phase, the relevant (non massive) fermionic particles (the electron and the neutrino) consist in a right-handed [32] electron $\mathsf{R} = e_R$ in an (weak) isospin singlet $I_W = 0$ and an isospin doublet $I_W = \frac{1}{2}$ made of the left-hande electron and the unique (left-handed) neutrino $\mathsf{L} = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix}$. The bosons are all non massive. The charges carried by the leptons follow from their weak isospin component $I_W^3$ and their hypercharge $Y$,

$$Q = I_W^3 + \frac{Y}{2}. \tag{150}$$

The hypercharge of the doublet is thus $Y_L = -1$ and that of the singlet is $Y_R = -2$. Under the non-Abelian weak isospin gauge transformation $SU(2)_W$, the fields change according to

$$\mathsf{R} \underset{SU(2)_W}{\longrightarrow} \mathsf{R}, \tag{151}$$

$$\mathsf{L} \underset{SU(2)_W}{\longrightarrow} \exp\left(\tfrac{1}{2}ig\boldsymbol{\sigma}\boldsymbol{\alpha}\right) \mathsf{L}, \tag{152}$$

and under the Abelian $U(1)_Y$ symmetry, they become

$$\mathsf{R} \underset{U(1)_Y}{\longrightarrow} \exp(-ig'\beta)\mathsf{R}, \tag{153}$$

$$\mathsf{L} \underset{U(1)_Y}{\longrightarrow} \exp(-ig'\beta/2)\mathsf{L}. \tag{154}$$

Note that the isospin coupling is $g$ while the hypercharge coupling is conventionally called $g'/2$.

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31. See Ryder pp307-312
32. In the Dirac Lagrangian $i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$, the right-handed and left-handed spinors are defined as $\mathsf{R} = \frac{1}{2}(1+\gamma_5)\psi$ and $\mathsf{L} = \frac{1}{2}(1-\gamma_5)\psi$. Since $\gamma_5$ and $\gamma_\mu$ commute, it follows that $i\bar{\psi}\gamma^\mu\partial_\mu\psi = i\bar{\mathsf{L}}\gamma^\mu\partial_\mu\mathsf{L} + i\bar{\mathsf{R}}\gamma^\mu\partial_\mu\mathsf{R}$.

$SU(2)_W \times U(1)_Y$ is made a local gauge symmetry through the introduction of gauge fields $\boldsymbol{W}_\mu$ and $X_\mu$ with the covariant derivative

$$\begin{aligned} \mathcal{D}_\mu \mathsf{L} &= \partial_\mu \mathsf{L} + \tfrac{1}{2} i g \boldsymbol{\sigma} \boldsymbol{W}_\mu \mathsf{L} - \tfrac{1}{2} i g' X_\mu \mathsf{L}, & (155) \\ \mathcal{D}_\mu \mathsf{R} &= \partial_\mu \mathsf{R} - i g' X_\mu \mathsf{R}, & (156) \end{aligned}$$

where $\boldsymbol{W}_\mu$ is a weak triplet gauge (non-massive) boson $I_W = 1$ with hypercharge zero and $X_\mu$ is also a non-massive boson which has zero hypercharge, but is in an isospin singlet $I_W = 0$.

If we forget about the pure gauge field contributions, the kinetic part of the Lagrangian [33] in the minimal coupling is given by Dirac Lagrangian (the leptonic particles are fermions with spin $\frac{1}{2}$) i.e.

$$\mathcal{L} = i\bar{\mathsf{R}}\gamma^\mu \left(\partial_\mu - i g' X_\mu\right) \mathsf{R} + i\bar{\mathsf{L}}\gamma^\mu \left(\partial_\mu + \tfrac{1}{2} i g \boldsymbol{\sigma} \boldsymbol{W}_\mu - \tfrac{1}{2} i g' X_\mu\right) \mathsf{L} \qquad (157)$$

The weakness of the gauge invariant formulation is obviously that it contains 4 massless gauge fields, while Nature (at the present energy scales) has only one, and that the fermions are similarly all non massive (if the electron would have a non zero mass in this theory, the corresponding neutrino would share the same mass, since it appears as the second component of an isospin doublet). The spontaneous symmetry breaking scenario leads to 3 massive gauge fields and at the same time, the electron acquires mass as well (but not the neutrino !).

### □ The Higgs mechanism

The symmetry is broken by introduction of a complex Higgs field

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \theta_1 + i\theta_2 \\ \theta_3 + i\theta_4 \end{pmatrix} . \qquad (158)$$

This is an isospin doublet $I_W = \frac{1}{2}$ with hypercharge unity $Y_\phi = 1$,

$$\mathcal{D}_\mu \phi = \left(\partial_\mu + \tfrac{1}{2} i g \boldsymbol{\sigma} \boldsymbol{W}_\mu + \tfrac{1}{2} i g' X_\mu\right) \phi, \qquad (159)$$

and a Lagrangian of the form

$$\mathcal{L}_{\text{Higgs}} = \mathcal{D}_\mu \phi^\dagger \mathcal{D}^\mu \phi - m^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 + \text{interaction with leptons.} \qquad (160)$$

33. We consider non massive fermions, otherwise a term like $m^2 \bar{\mathsf{L}}\mathsf{L}$ would assign the same mass to the electron and the neutrino.

The potential $V(|\phi|) = m^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2$ is chosen such that it gives rise to spontaneous symmetry breaking with $|\phi|^2 = -m^2/2\lambda = v/\sqrt{2}$. For the classical field, the choice $\theta_3 = v$ is made and a local gauge transformation eliminates the other $\theta_i$'s. Fluctuations around $v$ are introduced through

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} . \tag{161}$$

Acting with the covariant derivative gives

$$\mathcal{D}_\mu\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{2}ig(W^1_\mu - iW^2_\mu)(v + h(x)) \\ \partial_\mu h - \frac{1}{2}i(gW^3_\mu - g'X_\mu)(v + h(x)) \end{pmatrix} \tag{162}$$

and reported in the Lagrangian density, this leads to (up to cubic terms)

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} &= \tfrac{1}{2}\left[\partial_\mu h\partial^\mu h - \tfrac{1}{2}m^2(v + h(x))^2 - \tfrac{1}{4}\lambda(v + h(x))^4 \right. \\ &\quad \left. + \tfrac{1}{4}g^2v^2(W^1_\mu W^{\mu 1} + W^2_\mu W^{\mu 2}) + \tfrac{1}{4}(gW^3_\mu - g'X_\mu)(gW^{\mu 3} - g'X^\mu)v^2\right] \\ &= \tfrac{1}{2}\left[\partial_\mu h\partial^\mu h - \tfrac{1}{2}m^2(v + h(x))^2 - \tfrac{1}{4}\lambda(v + h(x))^4\right] \\ &\quad + M^2_W W^+_\mu W^{-\mu} + \tfrac{1}{2}M^2_Z Z_\mu Z^\mu, \end{aligned} \tag{163}$$

where the charged massive vector bosons are

$$W^\pm_\mu = (W^1_\mu \mp iW^2_\mu)/\sqrt{2} \tag{164}$$

with masses $M^2_W = \frac{1}{4}g^2v^2$ and the neutral massive boson is such that [34]

$$\begin{aligned} \tfrac{1}{2}M^2_Z Z_\mu Z^\mu &= \tfrac{1}{8}v^2(gW^3_\mu - g'X_\mu)(gW^{\mu 3} - g'X^\mu) \\ &= \tfrac{1}{8}v^2(W^{3*}_\mu, X^*_\mu) \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} \begin{pmatrix} W^{3\mu} \\ X^\mu \end{pmatrix} \\ &= \tfrac{1}{2}(Z_\mu{}^*, A^*_\mu) \begin{pmatrix} M^2_Z & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z^{3\mu} \\ A^\mu \end{pmatrix} . \end{aligned} \tag{165}$$

The last line is obtained by a diagonalization of the mass matrix by an orthogonal transformation

$$Z_\mu = \cos\theta_W W^3_\mu - \sin\theta_W X_\mu \tag{166}$$

$$A_\mu = \sin\theta_W W^3_\mu + \cos\theta_W X_\mu, \tag{167}$$

34. See Cheng and Li p351

and the masses of the neutral fields are

$$\begin{aligned} M_Z^2 &= \tfrac{1}{4} v^2 (g^2 + {g'}^2) && (168) \\ M_A^2 &= 0. && (169) \end{aligned}$$

The coupling constant of the (charged) leptons and the electromagnetic gauge field gets the value

$$e = g \sin\theta_W. \tag{170}$$

With the symmetry breaking scenario, the coupling between the Higgs fields and the leptons of the theory (Yukawa term which forms a Lorentz scalar by the coupling between a Dirac spinor with a scalar field) in

$$-G_e(\bar{\mathsf{R}}\phi^*\mathsf{L} + \bar{\mathsf{L}}\phi\mathsf{R}) \tag{171}$$

similarly leads to massive electrons [35]

$$m_e = G_e v/\sqrt{2}. \tag{172}$$

35. See Quigg p110